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INVERSE PROBLEM FOR A BURGERS TYPE PARABOLİC EQUATİON

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ABSTRACT

Wave processes are an effective means of transmitting energy and information. They are widely used in science and technology. Therefore, studying the propagation lans of various types of waves is an important anol rebvant tasu.
The paper considers the inverse problem of identifying the unknown coefficient on the right side of the Burgers types parabolic equation Direct problems for the Burgers equation are considered in the works of L. K.Martinson, J.Whitham and others. An additional condition for finding the unknown coefficient, which depends on the variable time, is given in integral form. A theorem on the uniqueness and stability of the solution was proved.

Introduction

The Burgers equation is a special case of the Navier-Stokes equation in the one-dimensional case and reflects both the effects of nonlinear propagation and the effects of diffusion. The Burgers equation arises when considering a wide class of processes in hydromechanics, nonlinear acoustics and plasma physics.

The Burgers equation with a source $F(x,t) \neq 0$ describes the dynamics of a physical system located in an external field and is a natural generalization of the homogeneous equation corresponding to autonorms notions.

An important result for the Burgers equation was oftained by Cole and Hoff, who showed that the nonlinear Burgers equation can be reduced into the linear heat conduction equation thraigh a change of variables.

In papers [1,2] (see also the bibliography in these papers) direct problems for the Burgers equation are considered.

Problem statement

Let $D = (0,1) \times (0,T], 0 < T = const$, the spaces $C^{l}(\cdot)$, $C^{l+\alpha}(\cdot)$, $C^{l,\frac{l}{2}}(\cdot), C^{l+\alpha,\frac{l+\alpha}{2}}(\cdot), l = 0,1,2, 0 < \alpha < 1$ and the norms in these spaces be defined as in [3, chapter 1]:

$$\|p\|_D^{(l)} = \sum_{k=0}^{l} \sup_D \left|\frac{\partial^k p(x,t)}{\partial x^k}\right|, \ \|q\|_T^{(l)} = \sum_{k=0}^{l} \sup_{[0,T]} \left|\frac{d^k q(t)}{dt^k}\right|.$$

We consider the following inverse problem on determining a pair of functions $\{f(t), u(x,t)\}$:

$$u_t - u_{xx} + uu_x = f(t)g(x), \ (x,t) \in D, (1)$$

$$u(x,0) = \varphi(x), \quad x \in [0,1], (2)$$

$$u(0,t) = \psi_0(t), \ u(1,t) = \psi_1(t), \ t \in [0,T], (3)$$

$$\int_0^1 u(x,t)dx = h(t), \ t \in [0,T], (4)$$

where $g(x), \varphi(x), \psi_0(t), \psi_1(t), h(t)$ are the given functions, $u_t = \frac{\partial u}{\partial t}, u_x = \frac{\partial u}{\partial x}, u_{xx} = \frac{\partial^2 u}{\partial x^2}$.

The coefficient inverse problem for a parabolic equation were studied in the papers [4,6].

Problem (1)-(4) belongs to the class of Hadamard ill-posed problems. Examples indicate that solutions to problem (1)-(4) may not always exist, and even if it they do, uniqueness or stability is not guaranteed. We will give an example of a violation of the correctness conditions - namely, an example of the instability of the solution. Let in problem (1)-(4) $n = 1$, $D = (0,1)$ and the input data have the following form : $g(x) \equiv g_{\,s}(x,t) = xe^{-st}$, $(x,t) \epsilon (0,1) \times (0,T)$, $\varphi_s(x) = x$, $x \epsilon [0,1]$, $\psi_{0s}(t) = 0$, $\psi_{1s}(t) = e^{-st}$, $h_s(t) = \frac{1}{2}e^{-st}, t \epsilon [0,T]$, s is a natural number.

It is easy to verify that the functions $\{f_s(t) = e^{-st} - s$, $u_s(x,t) = xe^{-st}\}$ for any s=1,2,… are exact solutions to problem (1)-(4) with the above input data. It is clear that by choosing natural numbers m and s (m≠s) , the differences $|g_m(x,t) - g_s(x,t)|$, $|\varphi_m(x) - \varphi_s(x)|$, $|\psi_{0m}(t) - \psi_{0s}(t)|$, $|\psi_{1m}(t) - \psi_{1s}(t)|$ $|h_m(t) - h_s(t)|$, that is , the perturbations of the input data can be made arbitrarily small in the norm C^l for any l=0,1,2,… however , in this case $|f_m(t) - f_s(t)| \geq |s - m|$, which shows the violation of the stability of the solution to the problem (1)-(4) .

Therefore, this problem should be treated proceeding from the general concepts of the theory of ill-posed problems.

We take the following assumptions for the data of problem (1)-(4):

1⁰. $g(x) \in C^{\alpha}[0,1]$, $\int_0^1 g(x)dx = g_0 \neq 0$;

2⁰. $\varphi(x) \in C^{2+\alpha}[0,1]$;

3⁰. $\psi_0(t), \psi_1(t) \in C^{1+\alpha}[0,T]$, $\varphi(0) = \psi_0(0), \varphi(1) = \psi_1(0)$;

4⁰. $h(t) \in C^{1+\alpha}[0,T]$.

Definition 1. The pair of functions $\{f(t), u(x,t)\}$ is called the solution of problem (1)-(4) if:

1) $f(t) \in C^{\alpha}[0,T]$;

2) $u(x,t) \in C^{2+\alpha,1+\alpha/2} \cap C(\overline{D})$;

3) the relations (1)-(4) are satisfied for these functions in an ordinary way.

Problem (1)-(4) is reduced to an equivalent problem.

Lemma 1. Let the initial data of problem (1)-(4) satisfy conditions 1^0-4^0 and $\int_0^1 \varphi(x)dx = h(0)$.

If the pair $\{f(t), u(x,t)\}$ is a solution to problem (1)-(4) in the sense of definition 1, then this pair is also a solution to problem (1), (2), (3),

$$f(t) = \left[h_t - u_x(1,t) + u_x(0,t) + \frac{\psi_1^2(t)}{2} - \frac{\psi_0^2(t)}{2}\right]/g_0, \quad t \in [0,T] \ (5)$$

and vice versa, the solution to problem (1), (2), (3), (5) in the sense of definition 1 is a solution to problem (1)-(4).

Proof. Let's assume $\{f(t), u(x,t)\}$ is a solution to the problem in the sense of definition 1. Integrate equation (1) in the interval (0,1) with respect to x. Taking into account the conditions of lemma 1, we obtain

$$\int_0^1 u_t dx - \int_0^1 u_{xx} dx + \int_0^1 u u_x dx = \int_0^1 f(t)g(x)dx \ (6)$$

From here we obtain formula (5).

Now let's assume that the pair $\{f(t), u(x,t)\}$ is a solution to problem (1), (2), (3), (5) in the sense of definition 1. Let's show that condition (4) is satisfied. In the ratio (6) instead of $f(t)$, we write the value of (5), and denote $\theta(t) = \int_0^1 u(x,t)dx - h(t)$, we obtain

$$\theta_t(t) = 0, \quad \theta(0) = 0.$$

It is clear that the only solution to the Cauchy problem for $\theta(t)$ is $\theta(t) = 0$. Hence it follows:

$$\int_0^1 u(x,t)dx - h(t) = 0.$$

The Lemma 1 is proved. Let us call problem (1), (2), (3) and (5) problem A. On the uniqueness and "conditional" stability of a solution the uniqueness theorem and the estimate of stability for the solution of the inverse problem occupy a central place in investigation of their well-posedness.

Define the following set:

$$K_\alpha = \{(f,u)/f(t) \in C^{\alpha}[0,T], u(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{D}), (x,t) \in \overline{D} \text{ for all } (f,u) \ \ |f(t)| \le c_1, t \in [0,T], |u|, |u_x|, |u_{xx}| \le c_2, (x,t) \in \overline{D}, 0 < c_1, c_2 = const\}$$

Let us assume that the two input sets $\{g_i(x), \varphi_i(x), \psi_{0i}(t), \psi_{1i}(t), h_i(t)\}, i = 1,2$.

Problem A for brevity of the further exposition, Problem A_1 will call Problem A_2. Let $\{f_1(t), u_1(x,t)\}$and $\{f_2(t), u_2(x,t)\}$be solutions of problems A_1 and A_2, respectively.

Theorem 1. Let the following conditions hold the function $\{g_i(x), \varphi_i(x), \psi_{0i}(t),$ $\psi_{1i}(t), h_i(t)\}, i = 1,2$ satisfy conditions 1^0-4^0, respectively; 2) solutions of problems A_1 and A_2 exist in the sense of definition 1 and they belong to the set K_α.

Then there exists a $T^*(0 < T^* \leq T)$, such that for $(x,t) \in [0,1] \times [0,T^*] = \overline{D_*}$the solution of problem A is unique, and the stability estimate

$$\|u_1 - u_2\|_D^{(0)} + \|f_1 - f_2\|_T^{(0)} \leq c_3 \left[\|g_1 - g_2\|_{[0,1]}^{(0)} + \|\varphi_1 - \varphi_2\|_{[0,1]}^{(0)} + \right.$$

$$+\|\psi_{01} - \psi_{02}\|_T^{(1)} + \|\psi_{11} - \psi_{12}\|_T^{(1)} + \|h_1 - h_2\|_T^{(1)} \ (7)$$

is valid.

Proof. Consider the function

$$F(x,t) = \phi(x) + (1-x)[\psi_0(t) - \psi_0(0)] + x[\psi_1(t) - \psi_1(0)], (x,t) \in \overline{D}.$$

It's obvious that $F(x,t) \in C^{2+\alpha,1+\alpha/2}(\overline{D})$ and

$$F(x,0) = \phi(x), x \in [0,1], F(0,t) = \psi_0(t), F(1,t) = \psi_1(t), t \in [0,T].$$

Denote

$$z(x,t) = u_1(x,t) - u_2(x,t), \lambda(t) = f_1(t) - f_2(t),$$
$$\delta_1(x) = g_1(x) - g_2(x), \delta_2(x) = \varphi_1(x) - \varphi_2(x),$$
$$\delta_3(t) = \psi_{01}(t) - \psi_{02}(t), \delta_4(t) = \psi_{11}(t) - \psi_{12}(t),$$
$$\delta_5(x) = h_1(t) - h_2(t), \delta_6(x,t) = F_1(x,t) - F_2(x,t),$$
$$w(x,t) = z(x,t) - \delta_6(x,t).$$

Subtracting relations from the relations of problem A1, the corresponding relations of problem A2, we obtain the problem of determining a pair $\{\lambda(t), w(x,t)\}$:

$$w_t - w_{xx} = \lambda(t) g_1(x) + \Phi(x,t), \ (x,t) \in D, (8)$$

$$w(x,0) = 0, x \in [0,1], (9)$$

$$w(0,t) = w(1,t) = 0, t \in [0,T], (10)$$

$$\lambda(t) = [z_x(0,t) - z_x(1,t)]/g_{0,1} + H(t), \ t \in [0,T], (11)$$

where

$$\Phi(x,t) = f_2(t)\delta_1(x) - z(x,t)u_{1x}(x,t) - z_x(x,t)u_2(x,t) - \delta_{6t}(x,t) + \delta_{6xx}(x,t),$$
$$H(t) = \{[\delta_{5t}(t) + \delta_4(t)(\psi_{11}(t) + \psi_{12}(t))/2 - \delta_3(t)(\psi_{01}(t) + \psi_{02}(t)/2]g_{01-}$$
$$- \delta_1(x)[h_{2t}(t) - u_{2x}(1,t) + u_{2x}(0,t) + \psi_{12}^2(t)/2 - \psi_{02}^2(t)/2]\}/g_{01}g_{0x}.$$

Under the conditions of theorem 1 and from the definition of the set K_α it follows that right the side of equation (8) satisfies the Holder condition. It means, there exists a classical solution of the problem defining of $w(x,t)$ from the conditions (8), (9), (10) and it can be represented in the form [3, p.468]:

$$w(x,t) = \int_0^t \int_0^1 G(x,t,\xi,\tau)[\lambda(\tau)g_1(\xi) + \Phi(\xi,\tau)]d\xi d\tau, (12)$$

where$G(x,t,\xi,\tau)$ is a fundamental solution to equation $u_t - u_{xx} = 0$, for which the following estimations [3, chapter IV] are true:

$$\int_0^1 \left|G_x^l(x,t,\xi,\tau)\right| d\xi \leq c_4 (t-\tau)^{-(l-\alpha)/2}, l = 0,1 \ (13)$$

where $0 < c_4 = const$.

Taking into account that $w(x,t) = z(x,t) - \delta_6(x,t)$, from (12) we will obtain:

$$(x,t) = \delta_6(x,t) + \int_0^t \int_0^1 G(x,t,\xi,\tau)[\lambda(\tau)g_1(\xi) + \Phi(\xi,\tau)]d\xi d\tau, (14)$$

Assume

$$\chi = \|u_1 - u_2\|_D^{(0)} + \|f_1 - f_2\|_T^{(0)}.$$

Under the conditions of theorem 1 and from definition of the set K_α, taking into account estimation (13) we will obtain:

$$|z(x,t)| \le c_5 \left[\|\delta_1\|_{[0,1]}^{(0)} + \|\delta_6\|_D^{(2,1)}\right] + c_6\chi t^{1+\frac{\alpha}{2}} + c_7\|z_x\|_D^{(0)} t^{1+\frac{\alpha}{2}},\ (x,t) \in \overline{D}, (15)$$

$$|\lambda(t)| \le c_8 \left[\|\delta_1\|_{[0,1]}^{(0)} + \|\delta_3\|_T^{(0)} + \|\delta_4\|_T^{(0)} + \|\delta_5\|_T^{(0)}\right] + c_9\|z_x\|_D^{(0)}, \quad t \in [0,T]\ (16)$$

Function $z_x(x,t)$ is estimated as follow

Under the conditions of Theorem 1, taking into account (15) from the last equality we obtain

$$\|z_x(x,t)\| \le c_{10} \left[\|\delta_1\|_{[0,1]}^{(0)} + \|\delta_6\|_{[0,1]}^{(2,1)}\right] + c_{11}\chi t^{1+\frac{\alpha}{2}} + c_{12}\|z_x\|_D^{(0)} t^{1+\frac{\alpha}{2}},\ (x,t) \in \overline{D}, (17)$$

where $c_{10}, c_{11}, c_{12} > 0$ depends on data of problem A1 and A2, the set K_α.

Inequalities (17) are satisfied at one value $(x,t) \in \overline{D}$. Therefore they must be satisfied also for maximum values of the left parts.

$$\|z_x\|_D^{(0)} \le c_{10} \left[\|\delta_1\|_{[0,1]}^{(0)} + \|\delta_6\|_D^{(2,1)}\right] + c_{11}\chi t^{1+\alpha/2} + c_{12}\|z_x\|_D^{(0)} t^{1+\alpha/2}. (18)$$

Let $T_1 (0 < T_1 \le T)$ be such a number, that $c_{12}T_1^{1+\alpha/2} < 1$. Then from (18) we have

$$\|z_x\|_D^{(0)} \le c_{13} \left[\|\delta_1\|_{[0,1]}^{(0)} + \|\delta_6\|_D^{(2,1)}\right] + c_{14}\chi t^{1+\alpha/2}.$$

Substituting the last inequality into (15) and (16), we obtain:

$$|z(x,t)| \le c_{15} \left[\|\delta_1\|_{[0,1]}^{(0)} + \|\delta_6\|_D^{(2,1)}\right] + c_{16}\chi t^{1+\alpha/2},\ (x,t) \in \overline{D}$$

$$|\lambda(t)| \le c_{17} \left[\|\delta_1\|_{[0,1]}^{(0)} + \|\delta_6\|_D^{(2,1)} + \|\delta_5\|_T^{(0)}\right] + c_{18}\chi t^{1+\alpha/2},\ t \in [0,T].$$

or

$$|z(x,t)| + |\lambda(t)| \le c_{19} \left[\|\delta_1\|_{[0,1]}^{(0)} + \|\delta_6\|_D^{(2,1)} + \|\delta_5\|_T^{(0)}\right] + c_{18}\chi t^{1+\alpha/2}.$$

The last inequality is satisfied for all $(x,t) \in \overline{D}$. Therefore it is also satisfied for the maximum values of the left points.

$$\chi \le c_{19} \left[\|\delta_1\|_{[0,1]}^{(0)} + \|\delta_6\|_D^{(2,1)} + \|\delta_5\|_T^{(0)}\right] + c_{18}\chi t^{1+\alpha/2}.$$

Let $T_2 (0 < T_2 \le T) R$ be such a number, that $c_{18}T_1^{1+\alpha/2} < 1$. By choosing $T^* = min(T_1, T_2)$ from the last inequality we get the estimate (7).

The uniqueness of the solution to problem A is obtained from formula (7) at

$$g_1(x) = g_2(x), \varphi_1(x) = \varphi_2(x)$$

$$\psi_{01}(t) = \psi_{02}(t)\ \psi_{11}(t) = \psi_{12}(t)\, h_1(t) = h_2(t).$$

Theorem 1 is proved.

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