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COEFFICIENT BOUNDS, INVESTIGATING INCLUSION PROPERTIES FOR A GENERAL SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY NOOR INTEGRAL OPERATOR

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ABSTRACT

In this paper, we introduce a new subclass $\mathrm{B}_m^*(\varphi,\delta,\alpha)$ of analytic functions defined via the Noor integral operator. The class is formulated through appropriate analytic conditions involving the parameters φ,δ and α. A systematic investigation of this subclass is presented, with particular emphasis on coefficient estimates for functions belonging to $\mathrm{B}_m^*(\varphi,\delta,\alpha)$. In addition, several closure theorems are established, demonstrating that the class is preserved under relevant algebraic operations. Fundamental structural properties of the proposed subclass are derived and discussed in detail. The obtained results extend and complement existing studies in geometric function theory and related subclasses of analytic functions.

1. INTRODUCTION

Let A represent the set of analytic functions f defined on the open unit disk $\mathrm{U}=\{z\in\mathbb{C}:|z|<1\}$, which are normalized by the conditions $f(0)=0$ and $f'(0)=1$. Consequently, every function $f\in\mathrm{A}$ can be expressed as a Taylor Maclaurin series of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\quad (z\in U). \tag{1}$$

In 1975, Silverman [1] introduced and analyzed a specific subset of comprised A of functions where the coefficients, starting from the second term, are negative. In other words, the analytic functions f within this subset can be represented as

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U), \ a_n \geq 0. \tag{2}$$

This subclass is known as the class of analytic functions with negative coefficients and is denoted as A^*. Following Silverman's contribution, there has been considerable interest in studying functions with negative coefficients. Building on Silverman 's work, many additional subclasses A of have been investigated in academic literature.

Denote by

$$\mathrm{R}^m := \frac{z}{(1-z)^{m+1}} * f(z) \quad (m \in \mathbb{N} \cup \{0\}).$$

Then implies that

$$\mathrm{R}^m f(z) = \frac{z\left(z^{n-1} f(z)\right)^m}{m!} \quad (m \in \mathbb{N} \cup \{0\}).$$

The operator $\mathrm{R}^m f$ is called Ruscheweyh derivative operator [2]. Noor [3] defined and investigated an integral operator $\mathrm{N}^m : \mathrm{A} \to \mathrm{A}$ analogous to $\mathrm{R}^m f$ as follows.

Let $f_m(z) = \dfrac{z}{(1-z)^{m+1}}, \ m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},$ and $f_m^{(-1)}$ be defined such that

$$f_m(z) * f_m^{(-1)}(z) = \frac{z}{1-z}.$$

Then

$$\mathrm{N}^m f(z) = f_m^{(-1)}(z) * f(z) = \left[\frac{z}{(1-z)^{m+1}}\right]^{(-1)} * f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(m+1)n!}{\Gamma(m+n)} a_n z^n, \tag{3}$$

where Γ is Euler gamma function.

Yousef et al. in [4] initially introduced the class $\mathrm{B}_{\Sigma}^{\mu}(\varphi, \delta, \alpha)$.

Definition 1: For $\varphi \geq 1, \delta \geq 0, \mu \geq 0$ and $0 \leq \alpha < 1$, a function $f \in \mathrm{A}$ represented by (1) is said to be in the class $\mathrm{B}_{\Sigma}^{\eta}(\varphi, \mu, \alpha)$ if the following conditions hold for all $z \in U.$

$$\mathrm{Re}\left\{(1-\varphi)\left(\frac{f(z)}{z}\right)^{\eta} + \varphi f'(z)\left(\frac{f(z)}{z}\right)^{\eta-1} + \xi \delta z f''(z)\right\} > \alpha,$$

where $\xi = \dfrac{2\delta + \eta}{2\lambda + 1}$.

Several researches have explored the class $\mathrm{B}_{\Sigma}^{\eta}(\varphi, \mu, \alpha)$ and utulized it in various contexts. For example, we refer the reader to [5-10].

Definition 2. For $\varphi \geq 1$, $\delta \geq 0$, and $0 \leq \alpha < 1$, an analytic function f represented by (2) is said to be in the class $\mathrm{B}_m^*(\varphi,\delta,\alpha)$ if the following conditions hold for all $z \in U$..

$$\operatorname{Re}\left\{(1-\varphi)\frac{\mathrm{N}^m f(z)}{z} + \varphi\left(\mathrm{N}^m f(z)\right)' + \delta z\left(\mathrm{N}^m f(z)\right)''\right\} > \alpha, \tag{4}$$

where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2. COEFFICIENTS ESTIMATES

This section commences with deriving a necessary and sufficient condition for a function $f(z)$ to belong to the class $\mathrm{B}_m^*(\varphi,\delta,\alpha)$.

Theorem 2.1. A function $f \in \mathrm{A}^*$ defined by (2) belong to the class $\mathrm{B}_m^*(\varphi,\delta,\alpha)$ if and only if

$$\sum_{n=2}^{\infty}\left[(1-\varphi)+n\varphi+n(n-1)\delta\right]\frac{\Gamma(m+1)n!}{\Gamma(m+n)}a_n \leq 1-\alpha, \tag{5}$$

where $\lambda \geq \mu \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Proof. Let the function $f(z)$ defined by (2) be in the class $\mathrm{B}_m^*(\varphi,\delta,\alpha)$ and $z \in U$.. Then by definition

$$\begin{aligned}&\operatorname{Re}\left\{(1-\varphi)\frac{\mathrm{N}^m f(z)}{z} + \varphi\left(\mathrm{N}^m f(z)\right)' + \delta z\left(\mathrm{N}^m f(z)\right)''\right\}\\ &= \operatorname{Re}\left\{1+\sum_{n=2}^{\infty}[1-\varphi+n\varphi+n(n-1)\delta]\frac{\Gamma(m+1)n!}{\Gamma(m+n)}a_n z^{n-1}\right\} > \alpha.\end{aligned}$$

As z approaches 1^- along the real axis, inequality (2) follows.

Conversely, the inequality (5) holds, then $z \in U$., we have

$$\begin{aligned}&\left|(1-\varphi)\frac{\mathrm{N}^m f(z)}{z} + \varphi\left(\mathrm{N}^m f(z)\right)' + \delta z\left(\mathrm{N}^m f(z)\right)'' - 1\right|\\ &= \left|\sum_{n=2}^{\infty}[1-\varphi+n\varphi+n(n-1)\delta]\frac{\Gamma(m+1)n!}{\Gamma(m+n)}a_n z^{n-1}\right|\\ &\leq \sum_{n=2}^{\infty}[1-\varphi+n\varphi+n(n-1)\delta]\frac{\Gamma(m+1)n!}{\Gamma(m+n)}a_n \leq 1-\alpha.\end{aligned}$$

This yields $f \in \mathrm{B}_m^*(\varphi,\delta,\alpha)$.

Corollary 2.2. Let $f(z)$ defined by (2) be in the class $\mathrm{B}_m^*(\varphi,\delta,\alpha)$. Then

$$a_n \leq \frac{1-\alpha,}{\left[1-\varphi+n\varphi+n(n-1)\delta\right]\frac{\Gamma(m+1)n!}{\Gamma(m+n)}}, \quad n \geq 2.$$

The above inequality holds for the function

$$f(z) = z - \frac{1-\alpha}{\left[1-\varphi+n\varphi+n(n-1)\delta\right]\frac{\Gamma(m+1)n!}{\Gamma(m+n)}} z^n.$$

3. INCLUSION RELATIONS

This section investigates the inclusion properties of the subclass $\mathrm{B}_m^*(\varphi,\delta,\alpha)$. By examining various inclusion relations, we aim to understand how different subclasses relate to each other within this framework. The results presented offer insights into how certain classes are contained within others, contributing to a broader understanding of the hierarchical structure within the class of analytic functions under consideration.

Theorem 3.1. Let $0 \le \alpha_1 \le \alpha_2 < 1$. Then

$$\mathrm{B}_m^*(\varphi,\delta,\alpha_1) \supseteq \mathrm{B}_m^*(\varphi,\delta,\alpha_2). \qquad (6)$$

Proof. Consider the function $f(z)$ defined by equation (2) to belong to the class $\mathrm{B}_m^*(\varphi,\delta,\alpha_2)$. Then, by Theorem 2.1, we have

$$\sum_{n=2}^{\infty}\left[(1-\varphi)+n\varphi+n(n-1)\delta\right]\frac{\Gamma(m+1)n!}{\Gamma(m+n)}a_n$$
$$\le 1-\alpha_2 \le 1-\alpha_1$$

Thus, $f \in \mathrm{B}_m^*(\varphi,\delta,\alpha_1)$.

Therefore, the inclusion relation shown by (6) holds.

Theorem 3.2. Let $1 \le \varphi_1 \le \varphi_2$. Then

$$\mathrm{B}_m^*(\varphi_1,\delta,\alpha) \supseteq \mathrm{B}_m^*(\varphi_2,\delta,\alpha). \qquad (7)$$

Proof. Consider the function $f(z)$ defined by equation (2) to belong to the class $\mathrm{B}_m^*(\varphi_2,\delta,\alpha)$. Then, by Theorem 2.1, we have

$$\sum_{n=2}^{\infty}\left[(1-\varphi)+n\varphi+n(n-1)\delta\right]\frac{\Gamma(m+1)n!}{\Gamma(m+n)}a_n$$
$$\le \sum_{n=2}^{\infty}\left[(1-\varphi_2)+n\varphi_2+n(n-1)\delta\right]\frac{\Gamma(m+1)n!}{\Gamma(m+n)}a_n$$
$$\le 1-\alpha.$$

Thus, $f \in \mathrm{B}_m^*(\varphi_1,\delta,\alpha)$.

Therefore, the inclusion relation shown by (7) holds.

Theorem 3.3. let $0 \le \delta_1 \le \delta_2$. Then

$$\mathrm{B}_m^*(\varphi,\delta_1,\alpha) \supseteq \mathrm{B}_m^*(\varphi,\delta_2,\alpha). \qquad (8)$$

Proof. Consider the function $f(z)$ defined by equation (2) to belong to the class $\mathrm{B}_m^*(\varphi,\delta_2,\alpha)$. Then, by Theorem 2.1, we have

$$\begin{aligned}&\sum_{n=2}^{\infty}\left[(1-\varphi)+n\varphi+n(n-1)\delta_1\right]\frac{\Gamma(m+1)n!}{\Gamma(m+n)}a_n\\&\leq\sum_{n=2}^{\infty}\left[(1-\varphi)+n\varphi+n(n-1)\delta_2\right]\frac{\Gamma(m+1)n!}{\Gamma(m+n)}a_n\\&\leq 1-\alpha.\end{aligned}$$

Thus, $f\in \mathrm{B}_m^*(\varphi,\delta_1,\alpha)$.

Therefore, the inclusion relation shown by (8) holds.

Theorem 3.4. Let $0\leq m_1\leq m_2$. Then

$$\mathrm{B}_{m_1}^*(\varphi,\delta,\alpha)\supseteq \mathrm{B}_{m_2}^*(\varphi,\delta,\alpha) \tag{9}$$

Proof. Consider the function $f(z)$ defined by equation (2) to belong to the class $\mathrm{B}_{m_2}^*(\varphi,\delta,\alpha)$. Then, by Theorem 2.1, we have

$$\begin{aligned}&\sum_{n=2}^{\infty}\left[(1-\varphi)+n\varphi+n(n-1)\delta\right]\frac{\Gamma(m_1+1)n!}{\Gamma(m_1+n)}a_n\\&\leq\sum_{n=2}^{\infty}\left[(1-\varphi)+n\varphi+n(n-1)\delta\right]\frac{\Gamma(m_2+1)n!}{\Gamma(m_2+n)}a_n\\&\leq 1-\alpha.\end{aligned}$$

Thus, $f\in \mathrm{B}_{m_1}^*(\varphi,\delta,\alpha)$.

Therefore, the inclusion relation shown by (9) holds.

4. CONCLUSION

In this study, a new subclass $\mathrm{B}_m^*(\varphi,\delta,\alpha)$ of analytic functions associated with the Noor integral operator has been introduced and examined. Several fundamental properties of this subclass have been established, including coefficient bounds and closure results. The findings contribute to the ongoing development of geometric function theory by extending existing results related to integral operators. The results obtained in this study are expected to provide a basis for further studies through the development of new function classes or integral operators.

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