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DIFFERENTIAL SUBORDINATION ASSOCIATED WITH GENERALIZED BESSEL-MAITLAND FUNCTION

MERVE KARAMAN[1], MURAT ÇAĞLAR[1,*]

[1]Department of Mathematics, Faculty of Science, Erzurum Technical University, Erzurum, Türkiye

degerlikaraman3661@gmail.com; murat.caglar@erzurum.edu.tr

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ABSTRACT

In this study, we introduce a new linear operator associated with the generalized Bessel–Maitland function, which is defined and analyzed in the open unit disk. The operator is constructed in such a way that it generalizes several well-known operators previously studied in geometric function theory. By employing the framework of differential subordination, we derive several results concerning analytic functions that are subordinate to or dominated by functions involving this operator. To achieve these results, we investigate suitable classes of admissible functions that guarantee the validity of the corresponding subordination relations. The findings presented in this work not only extend certain known results in the literature but also provide new insights into the geometric properties of analytic functions defined by the generalized Bessel–Maitland operator.

1. Introduction

The Let $\mathcal{H}(\mathbb{U})$ be the class of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$.

For $a \in \mathbb{C}$ and $n \in \mathbb{N} = \{1,2,\dots\}$ denote by $H[a,n]$ the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots.$$

The subclasses $\mathcal{H}[0,1]$ and $\mathcal{H}[1,1]$ will be simply denoted by $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively.

Let $\mathcal{A} \subset \mathcal{H}_0$ be the class of all functions f of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1}. \qquad (1.1)$$

Suppose that f and g are two members of $\mathcal{H}(\mathbb{U})$. We say that f is *subordinate* to g, denoted by $f \prec g$, if there exist a function $\omega \in \mathcal{H}(\mathbb{U})$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. If g is a univalent function in $\mathbb{U}$, then

$$f \prec g \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $f \in \mathcal{A}$ be given by (1.1) and let $g \in \mathcal{A}$ be given by

$$g(z) = z + \sum_{n=1}^{\infty} b_{n+1} z^{n+1}.$$

The Hadamart product (or convolution) of the functions f and g is defined by

$$(f * g)(z) = z + \sum_{n=1}^{\infty} a_{n+1} b_{n+1} z^{n+1} = (g * f)(z).$$

In the present perusal, we study some geometric properties of Generalized Bessel-Maitland function (see, [7], Eq.(8.3)), $J_{\eta}^{\gamma}(z)$. This function is defined by the following series representation:

$$J_{\eta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\Gamma(\gamma n+\eta+1)} \quad (\Re(\gamma) \geq 0, \Re(\eta) \geq -1 \text{ and } z \in \mathbb{U}). \quad (1.2)$$

It has many application in various research fields of Science an Engineering. For a comprehensive description of applications of Bessel functions and its generalization, the reader may be referred to [13]. Here in the present paper, we define a new (probably) generalization of Bessel-Maitland function called generalized Bessel-Maitland function $J_{\eta,a}^{\gamma}(z)$ given by:

$$J_{\eta,a}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(-a)^n z^n}{n!\Gamma(\gamma n+\eta+1)} \quad (a \in \mathbb{C} - \{0\}, \gamma > 0, \eta \geq -1 \text{ and } z \in \mathbb{U}). \quad (1.3)$$

It can be easily that

$$J_{\eta,-1}^{\gamma}(z) = W_{\gamma,\eta+1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\gamma n+\eta+1)} \quad (1.4)$$

where $W_{\gamma,\eta+1}(z)$ is called Wright function and

$$J_{\eta,1}^{\gamma}(z) = J_{\eta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\Gamma(\gamma n+\eta+1)}. \quad (1.5)$$

Observe that the Generalized Bessel-Maitland function $J_{\eta,a}^{\gamma}(z) \notin \mathcal{A}$. We can consider the following two types of normalization of the Generalized Bessel-Maitland function:

$$\mathbb{J}_{\eta,a}^{\gamma}(z) = z\Gamma(\eta+1) J_{\eta,a}^{\gamma}(z) = z + \sum_{n=1}^{\infty} \frac{(-a)^n \Gamma(\eta+1) z^{n+1}}{n!\Gamma(\gamma n+\eta+1)} \quad (1.6)$$

and

$$\mathcal{J}_{\eta,a}^{\gamma}(z) = \frac{\Gamma(\gamma+\eta+1)}{(-a)} \left(J_{\eta,a}^{\gamma}(z) - \frac{1}{\Gamma(\eta+1)} \right) = \sum_{n=0}^{\infty} \frac{(-a)^n \Gamma(\gamma+\eta+1) z^{n+1}}{(n+1)!\Gamma(\gamma n+\gamma+\eta+1)}. \quad (1.7)$$

$$(\gamma > 0, \gamma + \eta > -1, a \in \mathbb{C} - \{0\}, z \in \mathbb{U})$$

Also note that

$$\mathbb{J}_{\eta,1}^{1}(z) = \mathbb{J}_{\eta}(z) = \Gamma(\eta+1) z^{1-\eta/2} J_{\eta}(2\sqrt{z}) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(\gamma+\eta+1) z^{n+1}}{n!\Gamma(n+\eta+1)} \quad (1.8)$$

where $J_{\eta}(z)$ is well known Bessel function of order η and $\mathbb{J}_{\eta}(z)$ is the normalized Bessel function, studied recently for the various geometric properties (see [9]-[11]). Conversely, it can be easily seen that

Additionally, we observe that

$$\mathbb{V}_{\eta,a}^{\gamma}(z) = \frac{\mathbb{J}_{\eta,a}^{\gamma}(z)}{z} = \frac{1}{z}\left[z + \sum_{n=1}^{\infty} \frac{(-a)^n \Gamma(\eta+1) z^{n+1}}{n!\Gamma(\gamma n+\eta+1)} \right] = 1 + \sum_{n=1}^{\infty} \frac{(-a)^n \Gamma(\eta+1) z^n}{n!\Gamma(\gamma n+\eta+1)}$$

and

$$z\left(\mathbb{V}_{\eta,a}^{\gamma}(z)\right)' = \sum_{n=1}^{\infty} \frac{(-a)^n \Gamma(\eta+1) n z^n}{n!\Gamma(\gamma n+\eta+1)}.$$

The following identity relations can be easily establihed:

$$\gamma z \left(\mathbb{J}_{\eta+1,a}^{\gamma}(z)\right)' = (\eta+2)\mathbb{J}_{\eta+1,a}^{\gamma}(z) + (\gamma - \eta - 2)\mathbb{J}_{\eta+2,a}^{\gamma}(z), \quad (1.9)$$

$$z\left(\mathcal{J}_{\eta,a}^{\gamma}(z)\right)' = \mathbb{J}_{\gamma+\eta,a}^{\gamma}(z) \qquad (1.10)$$

and

$$\left(\mathbb{V}_{\eta,a}^{\gamma}(z)\right)' = \frac{(-a)\Gamma(\eta+1)}{\Gamma(\gamma+\eta+1)}\mathbb{V}_{\gamma+\eta,a}^{\gamma}(z). \qquad (1.11)$$

Differential subordination and superordination results for functions in $\mathcal{H}(\mathbb{U})$ involving various operators have been considered by [1]-[6],[10],[12].

In this paper, we define certain classes of admissible functions and obtain second-order differantial subordination results involving the operator $\mathbb{J}_{\eta,a}^{\gamma}(z)$.

In order to prove our results, the following definition and theorem will be required (see [8]).

Let Q denote the class of all functions $q(z)$ which are analytic and injective on $\overline{\mathbb{U}}\backslash E(q)$, where

$$E(q) = \left\{\zeta \in \partial\mathbb{U}: \lim_{z\to\zeta} q(z) = \infty\right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U}\backslash E(q)$. Further, let $Q(a)$ denotes the subclass of Q consisting of functions q for which $q(0) = a$. We write $Q_0 \equiv Q(0)$ and $Q_1 \equiv Q1)$.

Definition 1 *([8], Definition 2.3a, p. 27). Let Ω be a set in $\mathbb{C}$, $q \in Q$ and $n \in \mathbb{N}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi: \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\psi(r, s, t; z) \notin \Omega$$

whenever

$$r = q(\zeta), s = k\zeta q'(\zeta) \text{ and } \Re\left(\frac{t}{s}+1\right) \geqslant k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\}$$

with $z \in \mathbb{U}$, $\zeta \in \partial\mathbb{U}\backslash E(q)$ and $k \geqslant n$.

The class $\Psi_1[\Omega, q]$ will be simply denoted by $\Psi\Omega, q]$.

Theorem 1. *([8], Theorem 2.3b, p. 28) Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If the analytic function*

$$p(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots$$

satisfies

$$\psi(p(z), zp'(z), z^2p''(z);\ z) \in \Omega, \text{for all } z \in \mathbb{U}$$

then $p(z) \prec q(z)$.

2. Differential subordination involving the operator $\mathbb{J}_{\eta,a}^{\gamma}(z)$

First, we define a class of admissible functions which is needed to prove te second-order differential subordination theorem involving the operator $\mathbb{J}_{\eta,a}^{\gamma}(z)$ defined by (1.6).

Definition 2. *Let Ω be a set in $\mathbb{C}$ and $q(z) \in Q_0 \cap \mathcal{H}_0$. The class of admissible functions $\Phi_H[\Omega, q]$ consists of those functions $\phi: \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u, v, w; z) \notin \Omega$$

whenever

$$u = q(\zeta), v = \frac{\gamma k\zeta q'(\zeta)-(\eta+2)q(\zeta)}{\gamma-\eta-2} \ (\gamma > 0, \gamma+\eta > -1)$$

and

$$\Re\left\{\frac{(\gamma-\eta-3)w-(\eta+2)(\eta+3)u}{\gamma[(\gamma-\eta-2)v+(\eta+2)u]}-\frac{2(\gamma-\eta)-5}{\gamma}\right\}\geq k\Re\left\{\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right\}$$

$$(z\in\mathbb{U};\ \gamma>0;\eta\in\partial\mathbb{U}\backslash E(q);\ k\geq 1).$$

Theorem 2. *Let* $\phi\in\Phi_H[\Omega,q]$. *If* $f\in\mathcal{A}$ *satisfies the following inclusion relationship:*

$$\left\{\phi\left(\mathbb{J}^{\gamma}_{\eta+1,a}(z),\mathbb{J}^{\gamma}_{\eta+2,a}(z),\mathbb{J}^{\gamma}_{\eta+3,a}(z);z\right):z\in\mathbb{U}\right\}\subset\Omega, \qquad (2.1)$$

then

$$\mathbb{J}^{\gamma}_{\eta+1,a}(z)\prec q(z)\ (z\in\mathbb{U}).$$

Proof. We define the analytic function p in $\mathbb{U}$ by

$$p(z)=\mathbb{J}^{\gamma}_{\eta+1,a}(z). \qquad (2.2)$$

Then, differentiating (2.2) with respect to z and using the recursive relation (1.9), we have

$$\mathbb{J}^{\gamma}_{\eta+2,a}(z)=\frac{\gamma zp'(z)-(\eta+2)p(z)}{\gamma-\eta-2}. \qquad (2.3)$$

Further computations show that

$$\mathbb{J}^{\gamma}_{\eta+3,a}(z)=\frac{\gamma^2z^2p''(z)+p'(z)\gamma z(\gamma-2\eta-5)+(\eta+3)(\eta+2)p(z)}{\gamma-\eta-3}. \qquad (2.4)$$

We now define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u=r,v=\frac{\gamma s-(\eta+2)r}{\gamma-\eta-2}\ and\ w=\frac{\gamma^2t+\gamma s(\gamma-2\eta-5)+(\eta+3)(\eta+2)r}{\gamma-\eta-3}.$$

Let

$$\psi(r,s,t;z)=\phi(u,v,w;z)$$

$$=\phi\left(r,\frac{\gamma s-(\eta+2)r}{\gamma-\eta-2},\frac{\gamma^2t+\gamma s(\gamma-2\eta-5)+(\eta+3)(\eta+2)r}{\gamma-\eta-3};z\right). \qquad (2.5)$$

Using equations (2.2)-(2.4), and from (2.5), we get

$$\psi(p(z),zp'(z),z^2p''(z);z)=\phi\left(\mathbb{J}^{\gamma}_{\eta+1,a}(z),\mathbb{J}^{\gamma}_{\eta+2,a}(z),\mathbb{J}^{\gamma}_{\eta+3,a}(z);z\right). \qquad (2.6)$$

Hence (2.1) assumes the following form

$$\psi(p(z),zp'(z),z^2p''(z);z)\in\Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi\in\Phi_H[\Omega,q]$ is equivalent to the admissibility condition for as given in Definition 1. Note that

$$\frac{t}{s}+1=\frac{w(\gamma-\eta-3)-(\eta+3)(\eta+2)u}{\gamma v(\gamma-\eta-2)+(\eta+2)u]}-\frac{2(\gamma-\eta)-5}{\gamma},$$

and hence $\psi\in\Psi[\Omega,q]$. By Theorem 1, we have

$$p(z)\prec q(z)\text{ or }\mathbb{J}^{\gamma}_{\eta+1,a}(z)\prec q(z)$$

which completes the proof of Theorem 2.

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case the class $\Phi_H[h(\mathbb{U}),q]$ is written as $\Phi_H[h,q]$.

The following result is an immediate consequence of Theorem 2.

Corollary 1. *Let $\phi \in \Phi_H[\Omega, q]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\mathbb{J}_{\eta+1,a}^{\gamma}(z), \mathbb{J}_{\eta+2,a}^{\gamma}(z), \mathbb{J}_{\eta+3,a}^{\gamma}(z); z\right) \prec h(z), \qquad (2.7)$$

then

$$\mathbb{J}_{\eta+1,a}^{\gamma}(z) \prec q(z)\ (z \in \mathbb{U}).$$

Our next result is an extension of Theorem 2 to the case when the behavior of q on $\partial\mathbb{U}$ is not known.

Corollary 2. *Let $\Omega \subset \mathbb{C}$ and let q be univalent in $\mathbb{U}$ with $q(0) = 0$. Let $\phi \in \Phi_H[\Omega, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \mathcal{A}$ satisfies the following inclusion relationship*

$$\phi\left(\mathbb{J}_{\eta+1,a}^{\gamma}(z), \mathbb{J}_{\eta+2,a}^{\gamma}(z), \mathbb{J}_{\eta+3,a}^{\gamma}(z); z\right) \in \Omega,$$

then

$$\mathbb{J}_{\eta+1,a}^{\gamma}(z) \prec q(z)\ (z \in \mathbb{U}).$$

Proof. We note from Theorem 2 that

$$\mathbb{J}_{\eta+1,a}^{\gamma}(z) \prec q_\rho(z)\ (z \in \mathbb{U}).$$

The result asserted by Corollary 2 is now deduced from the following subordination relationship

$$q_\rho(z) \prec q(z)\ (z \in \mathbb{U}).$$

Theorem 3. *Let h and q be univalent in $\mathbb{U}$ with $q(0) = 0$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi: \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ satisfy one of the following conditions*

1. $\phi \in \Phi_H[\Omega, q_\rho]$ for some $\rho \in (0,1)$, or

2. there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_H[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}$ satisfies (2.7), then

$$\mathbb{J}_{\eta+1,a}^{\gamma}(z) \prec q(z)\ (z \in \mathbb{U}).$$

Proof. The proof is similar to the proof of [8, Theorem 2.3d] and is therefore omitted.

The next theorem yields the best dominant of the differential subordination (2.7).

Theorem 4. *Let h be univalent in $\mathbb{U}$ and $\phi: \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{\gamma z q'(z) - (\eta+2)q(z)}{\gamma-\eta-2}, \frac{\gamma^2 z^2 q''(z) + \gamma(\gamma-2\eta-5) z q'(z) + (\eta+3)(\eta+2)q(z)}{\gamma-\eta-3}; z\right) = h(z) \qquad (2.8)$$

has a solution q with $q(0) = 0$ and satisfies one of the following conditions

1. $q \in \mathcal{Q}_0$ and $\phi \in \Phi_H[h, q]$.

2. q is univalent in $\mathbb{U}$ and $\phi \in \Phi_H[h, q_\rho]$ for some $\rho \in (0,1)$

3. q is univalent in $\mathbb{U}$ and there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_H[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.

If $f \in \mathcal{A}$ satisfies (2.7), then

$$\mathbb{J}_{\eta+1,a}^{\gamma}(z) \prec q(z)\ (z \in \mathbb{U})$$

and q is the best dominant.

Proof. Following the same arguments as in [8, Theorem 2.3e], we deduce that q is a dominant from Corollary 1 and Theorem 3. Since q satisfies (2.8), it is also a solution of (2.7) and therefore q will be dominated by all dominants. Hence q is the best dominant.

In the particular case when $q(z) = Mz$; $M > 0$, and in view of Definition 2, the class of admissible functions $\Phi_H[\Omega, q]$, denoted by $\Phi_H[\Omega, M]$, is described below.

Definition 3. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_H[\Omega, M]$ consists of those functions $\phi: \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ such that*

$$\phi\left(Me^{i\theta}, \frac{\gamma k-\eta-2}{\gamma-\eta-2}Me^{i\theta}, \frac{\gamma^2 L+[k\gamma(\gamma-2\eta-5)+(\eta+3)(\eta+2)]Me^{i\theta}}{\gamma-\eta-3}; z\right) \notin \Omega, \quad (2.9)$$

whenever $z \in \mathbb{U}, \gamma > 0$ and $\Re\left(Le^{-i\theta}\right) \geq k(k-1)M$ for all $\theta \in \mathbb{R}, k \geq 1$.

Corollary 3. *Let $\phi \in \Phi_H[\Omega, M]$. If $f \in \mathcal{A}$ satisfies the following inclusion relationship*

$$\phi\left(\mathbb{J}^{\gamma}_{\eta+1,a}(z), \mathbb{J}^{\gamma}_{\eta+2,a}(z), \mathbb{J}^{\gamma}_{\eta+3,a}(z); z\right) \in \Omega,$$

then

$$\mathbb{J}^{\gamma}_{\eta+1,a}(z) \prec Mz \ (z \in \mathbb{U}).$$

In the special case when $\Omega = \{w: |w| < M\} = q(\mathbb{U})$, the class $\Phi_H[\Omega, M]$ is simply denoted by $\Phi_H[M]$. Corollary 3 can now be written in the following form.

Corollary 4. *Let $\phi \in \Phi_H[M]$. If $f \in \mathcal{A}$ satisfies the following inequality*

$$\left|\phi\left(\mathbb{J}^{\gamma}_{\eta+1,a}(z), \mathbb{J}^{\gamma}_{\eta+2,a}(z), \mathbb{J}^{\gamma}_{\eta+3,a}(z); z\right)\right| < M,$$

then

$$\left|\mathbb{J}^{\gamma}_{\eta+1,a}(z)\right| < M.$$

Corollary 5 *If $M > 0, \gamma > 0$ and $f \in \mathcal{A}$ satisfies the following inequality*

$$\left|\mathbb{J}^{\gamma}_{\eta+2,a}(z)\right| < M,$$

then

$$\left|\mathbb{J}^{\gamma}_{\eta+1,a}(z)\right| < M.$$

Proof. This follows from Corollary 4 by taking $\phi(u, v, w; z) = v = \frac{\gamma k-\eta-2}{\gamma-\eta-2}Me^{i\theta}$.

Corollary 6. *Let $\gamma > 0$ and $M > 0$. If $f \in \mathcal{A}$ satisfies the following inequality*

$$\left|(\gamma-\eta-2)\mathbb{J}^{\gamma}_{\eta+2,a}(z) + (\eta+2)\mathbb{J}^{\gamma}_{\eta+1,a}(z)\right| < \frac{M}{|\gamma|}$$

then

$$\left|\mathbb{J}^{\gamma}_{\eta+2,a}(z)\right| < M.$$

Proof. Let $\phi(u, v, w; z) = \frac{\gamma-\eta-2}{\gamma}u + \frac{\eta+2}{\gamma}v$ and $\Omega = h(\mathbb{U})$ where $h(z) = \frac{Mz}{\gamma}$, $M > 0$. In order to use Corollary 3, we need to show that $\phi \in \Phi_H[\Omega, M]$, that is, the admissibility condition (2.9) is satisfied. This follows since

$$\left|\phi\left(Me^{i\theta},\frac{\gamma k-\eta-2}{\gamma-\eta-2}Me^{i\theta},\frac{\gamma^2 L+[k\gamma(\gamma-2\eta-5)+(\eta+3)(\eta+2)]Me^{i\theta}}{\gamma-\eta-3};z\right)\right|=\left|\frac{kMe^{i\theta}}{\gamma}\right|\geq\frac{M}{|\gamma|}$$

whenever $z\in\mathbb{U},\theta\in\mathbb{R},\gamma>0$ and $k\geq 1$. The result now follows from Corollary 3.

Theorem 4 shows that the result is sharp. The differential equation

$$zq'(z)=Mz$$

has a univalent solution $q(z)=Mz$. It follows from Theorem 4 that $q(z)=Mz$ is the best dominant.

Definition 4. *Let Ω be a set in $\mathbb{C}$ and $q\in\mathcal{Q}_1\cap\mathcal{H}_1$. The class of admissible functions $\Phi_{H,2}[\Omega,q]$ consists of those functions $\phi:\mathbb{C}^3\times\mathbb{U}\to\mathbb{C}$ that satisfy the admissibility condition*

$$\phi(u,v,w;z)\notin\Omega$$

where

$$u=q(\zeta),v=\frac{\gamma k\zeta q'(\zeta)+(\gamma-\eta-2)q(\zeta)}{\gamma-\eta-2}\ (\gamma>0,\gamma+\eta>-1)$$

and

$$\Re\left(\frac{(\gamma-\eta-3)(w-u)-(3\gamma-2\eta-5)(v-u)}{\gamma(\gamma-\eta-2)(\gamma-\eta-3)(v-u)}+1\right)\geq k\Re\left(\frac{\zeta q''(\zeta)}{q'(\zeta)}+1\right)$$
$$(z\in\mathbb{U};\gamma>0;\eta\in\partial\mathbb{U}\backslash E(q);k\geq 1).$$

Theorem 5. *Let $\phi\in\Phi_{H,2}[\Omega,q]$. If $f\in\mathcal{A}$ satisfies the following inclusion relationship*

$$\left\{\phi\left(\frac{\mathbb{J}^{\gamma}_{\eta+1,a}(z)}{z},\frac{\mathbb{J}^{\gamma}_{\eta+2,a}(z)}{z},\frac{\mathbb{J}^{\gamma}_{\eta+3,a}(z)}{z};z\right):z\in\mathbb{U}\right\}\subset\Omega,\quad(2.10)$$

then

$$\frac{\mathbb{J}^{\gamma}_{\eta+1,a}(z)}{z}\prec q(z)\ (z\in\mathbb{U}).$$

Proof. Let us define the analytic function p in $\mathbb{U}$ by

$$p(z)=\frac{\mathbb{J}^{\gamma}_{\eta+1,a}(z)}{z}.\quad(2.11)$$

By making use of (1.9) and (2.11), we get

$$\frac{\mathbb{J}^{\gamma}_{\eta+2,a}(z)}{z}=\frac{\gamma zp'(z)+(\gamma-\eta-2)p(z)}{\gamma-\eta-2}.\quad(2.12)$$

Further computations show that

$$\frac{\mathbb{J}^{\gamma}_{\eta+3,a}(z)}{z}=\frac{\gamma^2z^2p''(z)+\gamma(3\gamma-2\eta-5)zp'(z)+(\gamma-\eta-2)(\gamma-\eta-3)p(z)}{(\gamma-\eta-2)(\gamma-\eta-3)}.\quad(2.13)$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u=r,v=\frac{\gamma s+(\gamma-\eta-2)r}{\gamma-\eta-2}\text{ and }w=\frac{\gamma^2t+\gamma(3\gamma-2\eta-5)s+(\gamma-\eta-2)(\gamma-\eta-3)r}{(\gamma-\eta-2)(\gamma-\eta-3)}.\quad(2.14)$$

Let

$$\psi(r,s,t;z)=\phi(u,v,w;z)$$
$$=\phi\left(r,\frac{\gamma s+(\gamma-\eta-2)r}{\gamma-\eta-2},\frac{\gamma^2t+\gamma(3\gamma-2\eta-5)s+(\gamma-\eta-2)(\gamma-\eta-3)r}{(\gamma-\eta-2)(\gamma-\eta-3)};z\right).\quad(2.15)$$

The proof shall make use of Theorem 1. Using equations (2.11), (2.12) and (2.13), from (2.15) we obtain

$$\psi(p(z), zp'(z), z^2p''(z); z) = \phi\left(\frac{\mathbb{J}^{\gamma}_{\eta+1,a}(z)}{z}, \frac{\mathbb{J}^{\gamma}_{\eta+2,a}(z)}{z}, \frac{\mathbb{J}^{\gamma}_{\eta+3,a}(z)}{z}; z\right). \quad (2.16)$$

Hence (2.10) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\phi \in \Phi_{H,2}[\Omega, q]$ is equivalent to the admissibility condition for as given in Definition 1. Note that

$$\frac{t}{s} + 1 = \frac{(\gamma - \eta - 3)(w - u) - (3\gamma - 2\eta - 5)(v - u)}{\gamma(\gamma - \eta - 2)(\gamma - \eta - 3)(v - u)} + 1,$$

and hence $\psi \in \Psi\Omega, q]$. By Theorem 1, we have $p(z) \prec q(z)$ or

$$\frac{\mathbb{J}^{\gamma}_{\eta+1,a}(z)}{z} \prec q(z).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case the class $\Phi_{H,2}[h(\mathbb{U}), q]$ is written as $\Phi_{H,2}[h, q]$.

The following result is an immediate consequence of Theorem 5.

Corollary 7. *Let $\phi \in \Phi_{H,2}[h, q]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{\mathbb{J}^{\gamma}_{\eta+1,a}(z)}{z}, \frac{\mathbb{J}^{\gamma}_{\eta+2,a}(z)}{z}, \frac{\mathbb{J}^{\gamma}_{\eta+3,a}(z)}{z}; z\right) \prec h(z), \quad (2.17)$$

then

$$\frac{\mathbb{J}^{\gamma}_{\eta+1,a}(z)}{z} \prec q(z) \ (z \in \mathbb{U}).$$

In the particular case when $q(z) = 1 + Mz$; $M > 0$, the class of admissible functions $\Phi_{H,2}[\Omega, q]$, denoted by $\Phi_{H,2}[\Omega, M]$.

Definition 5. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,2}[\Omega, M]$ consists of those functions $\phi: \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ such that*

$$\phi\left(1 + Me^{i\theta}, 1 + \frac{(k+1)\gamma-\eta-2}{\gamma-\eta-2}Me^{i\theta}, 1 + \frac{\gamma^2 L + [\gamma k(3\gamma-2\eta-5) + (\gamma-\eta-2)(\gamma-\eta-3)]Me^{i\theta}}{(\gamma-\eta-2)(\gamma-\eta-3)}; z\right) \notin \Omega \quad (2.18)$$

whenever $z \in \mathbb{U}, \gamma > 0$ and $\Re\left(Le^{-i\theta}\right) \geq k(k-1)M$ for all $\theta \in \mathbb{R}, k \geq 1$.

Corollary 8. *Let $\phi \in \Phi_{H,2}[\Omega, M]$. If $f \in \mathcal{A}$ satisfies the following inclusion relationship*

$$\phi\left(\frac{\mathbb{J}^{\gamma}_{\eta+1,a}(z)}{z}, \frac{\mathbb{J}^{\gamma}_{\eta+2,a}(z)}{z}, \frac{\mathbb{J}^{\gamma}_{\eta+3,a}(z)}{z}; z\right) \in \Omega,$$

then

$$\frac{\mathbb{J}^{\gamma}_{\eta+1,a}(z)}{z} - 1 \prec Mz \ (z \in \mathbb{U}).$$

In the special case when $\Omega = \{w: |w - 1| < M\} = q(\mathbb{U})$, the class $\Phi_{H,2}[\Omega, M]$ is simply denoted by $\Phi_{H,2}[M]$ and Corollary 8 takes the following form.

Corollary 9. *Let* $\phi \in \Phi_{H,2}[M]$. *If* $f \in \mathcal{A}$ *satisfies the following inequality*

$$\left|\phi\left(\frac{\mathbb{J}_{\eta+1,a}^{\gamma}(z)}{z}, \frac{\mathbb{J}_{\eta+2,a}^{\gamma}(z)}{z}, \frac{\mathbb{J}_{\eta+3,a}^{\gamma}(z)}{z}; z\right) - 1\right| < M,$$

then for all $z \in \mathbb{U}$ we have

$$\left|\frac{\mathbb{J}_{\eta+1,a}^{\gamma}(z)}{z} - 1\right| < M.$$

Corollary 10. *Let* $\gamma > 0$ *and* $M > 0$. *If* $f \in \mathcal{A}$ *satisfies the following inequality*

$$\left|\frac{\mathbb{J}_{\eta+2,a}^{\gamma}(z)}{z} - \frac{\mathbb{J}_{\eta+1,a}^{\gamma}(z)}{z}\right| < \left|\frac{\gamma}{\gamma - \eta - 2}\right| M,$$

then for all $z \in \mathbb{U}$, we have

$$\left|\frac{\mathbb{J}_{\eta+1,a}^{\gamma}(z)}{z} - 1\right| < M.$$

Proof. This follows from Corollary 8 by taking $\phi(u, v, w; z) = v - u$.

7. Conclusion

In this study, a novel linear operator associated with the generalized Bessel–Maitland function has been introduced and investigated within the open unit disk. By applying the theory of differential subordination and admissible functions, several new subordination and domination results have been established for analytic functions involving this operator. The results obtained not only unify and generalize various existing operators and findings in geometric function theory but also reveal new aspects of the geometric behavior of analytic functions defined through the generalized Bessel–Maitland framework. These findings may serve as a foundation for further studies exploring additional subclasses of analytic or bi-univalent functions related to special functions and fractional calculus operators.

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