2024. Volume 8, Number 1 Pages 55-63

UOT:517.91./93 (517.95)
DOİ: https://doi.org/10.30546/09090.2024.110.206

GREEN'S FUNCTION OF AN IMPULSIVE STURM – LIOUVILLE OPERATOR

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ARTICLE INFO

Article history:
Received:2024-09-13
Received in revised form:2024-10-29
Accepted:2024-12-09
Available online

Keywords: Impulsive operators; Sturm – Liouville operators; Spectral singularities

ABSTRACT

This work is dedicated to the impulsive Sturm - Liouville operator on the whole axis with complex almost periodic potentials and the discontinuous coefficient on the right – hand side. We investigated the main characteristics of the fundamental solutions of the Sturm – Liouville equation. From the impulsive condition we found the transfer matrix. Using the impulsive condition and transfer matrix, we constructed Green's function and obtained the resolvent of the impulsive Sturm – Liouville operator. In future works, eigenvalues of the impulsive Sturm - Liouville operator will be investigated. The inverse problem will be formulated, a constructive procedure for the solution of the inverse problem will be provided.

1. Introduction

In the presented work, we consider the Sturm – Liouville equation on the whole axis

$$-y''+q(x)y=\lambda^2\rho(x)y,\quad x\in(-\infty,\infty)$$

where λ is a spectral parameter, and ρ is the density function.

In mathematical physics and quantum mechanics, boundary–value problems with discontinuities inside an interval are of great interest. To solve interior discontinuities, extra conditions, often called impulsive conditions, are imposed on the discontinuous point. The theory of impulsive differential equations was studied in detail in applied mathematics by Bainov and Simenov in 1995. Many authors have also studied the spectral theory of impulsive differential equations. Recently, the physical meaning and potential applications of spectral singularities of impulsive differential equations have been understood and studied by Mostafazadeh in 2011. Mostafazadeh, in his work, provided the physical meanings of eigenvalues and spectral singularities of the Schrödinger equation at a single point. In this work, we are concerned with the impulsive Sturm–Liouville operator on the whole axis, constructing Green's function and finding the resolvent.

2 Statement of the problem

Let's consider the Sturm – Liouville operator L in $L_2(-\infty,\infty)$ generated by the equation

$$-y''+q(x)y=\lambda^2\rho(x)y,\ \ x\in(-\infty,0)\cup(0,\infty) \tag{1}$$

with the impulsive condition

$$\begin{bmatrix} y\left(0^{+}\right) \\ y'\left(0^{-}\right) \end{bmatrix} = B \begin{bmatrix} y\left(0^{-}\right) \\ y'\left(0^{-}\right) \end{bmatrix}, \quad B = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \tag{2}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are complex numbers such that $\det B \neq 0$,

$$q(x) = \sum_{n=1}^{\infty} q_n e^{i\Lambda_n x} \tag{3}$$

and the condition

$$\sum_{n=1}^{\infty} |q_n| < \infty \tag{4}$$

is satisfied.

The set of exponents is a countable set of positive real numbers closed to the addition

$$M = \{\Lambda_1, \Lambda_2, \Lambda_3, ...\Lambda_n, ...\}, \ \Lambda_n > 0, \ n \in N. \tag{5}$$

The density function $\rho(x)$ has the form

$$\rho(x) = \begin{cases} 1, & x < 0 \\ \beta^2, & x > 0 \end{cases} \tag{6}$$

where $\beta > 0, \beta \neq 1$.

Furthermore, we denote the solutions of the equation (1) by y_- and y_+, respectively:

$$\begin{cases} y_-(x) := y(x), \ x < 0 \\ y_+(x) := y(x), \ x > 0 \end{cases} \tag{7}$$

Theorem 1. Equation (1) – (2) with potential $q(x)$ has the form of (3) and $\rho(x)$ defined as (6) has fundamental solutions of the form

$$f_1^{\pm}(x, \lambda) = e^{\pm i\lambda x} \left(1 + \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}}{\Lambda_n \pm 2\lambda} e^{i\Lambda_\alpha x} \right) \tag{8}$$

on $(-\infty, 0)$ satisfying the asymptotic condition

$$\lim_{\operatorname{Im} x \to -\infty} f_1^{\pm}(x, \lambda) e^{\mp i\lambda x} = 1 \qquad \text{for } \pm \operatorname{Im} \lambda > 0$$

On the other hand, equation (1) – (2) has the other fundamental solutions in the form of

$$f_2^{\pm}(x, \lambda) = e^{\pm i\beta\lambda x} \left(1 + \sum_{n=1}^{\infty} \sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}}{\Lambda_n \pm 2\beta\lambda} e^{i\Lambda_\alpha x} \right) \tag{9}$$

on the interval $(0,\infty)$ satisfying the asymptotic condition

$$\lim_{\mathrm{Im}\, x\to+\infty} f_2^{\pm}(x,\lambda)e^{\mp i\beta\lambda x}=1 \qquad \text{for } \pm\mathrm{Im}\lambda>0$$

here the numbers $V_{n\alpha}$ are determined from the following relations

$$\Lambda_\alpha(\Lambda_\alpha-\Lambda_n)V_{n\alpha}+\sum_{\beta\oplus\gamma=n}V_{n\beta}q_\gamma=0 \tag{10}$$

$$q_\alpha+\sum_{\beta\oplus\gamma=n}V_{n\beta}q_\gamma=0. \tag{11}$$

and series

$$\sum_{n=1}^{\infty}\Lambda_n^{-1}\sum_{\alpha=n}^{\infty}\Lambda_\alpha(\Lambda_\alpha-\Lambda_n)|V_{n\alpha}| \tag{12}$$

converges.

We easily see that at the points $\lambda=\mp\frac{\Lambda_n}{2}\left(\mp\frac{\Lambda_n}{2\beta}\right)$, $n\in N$ there can be simple poles to the function $f(x,\lambda)$.

Remark 1. If $\lambda\neq-\frac{\Lambda_n}{2}$ and $\mathrm{Im}\lambda<0$, then $f_1^+(x,\lambda)\in L_2(-\infty,0)$

Remark 2. If $\lambda\neq-\frac{\Lambda_n}{2\beta}$ and $\mathrm{Im}\lambda>0$, then $f_2^+(x,\lambda)\in L_2(0,\infty)$

Taking into account that the potential $q(x)$ can be extended to the upper semi – plane as an analytic function, we find

$$W\left[f_1^+(x,\lambda),\, f_1^-(x,\lambda)\right]=-2i\lambda \ \text{ for } \lambda\neq 0,\pm\frac{\Lambda_n}{2} \tag{13}$$

$$W\left[f_2^+(x,\lambda),\, f_2^-(x,\lambda)\right]=-2i\beta\lambda \ \text{ for } \lambda\neq 0,\pm\frac{\Lambda_n}{2\beta} \tag{14}$$

Therefore, the functions $f_1^+(x,\lambda),\, f_1^-(x,\lambda)\left(f_2^+(x,\lambda),\, f_2^-(x,\lambda)\right)$ are linearly independent solutions of the equation (1) for $\lambda\neq 0,\pm\frac{\Lambda_n}{2},\pm\frac{\Lambda_n}{2\beta}$.

Using linearly independent solutions of (1) in the intervals $(-\infty,0)$ and $(0,\infty)$, we can express the general solution of (1) by

$$\begin{cases} y_-(x,\lambda)=A_- f_1^+(x,\lambda)+B_- f_1^-(x,\lambda), & x<0 \\ y_+(x,\lambda)=A_+ f_2^+(x,\lambda)+B_+ f_2^-(x,\lambda), & x>0 \end{cases} \tag{15}$$

where $A_{\pm}$ and $B_{\pm}$ are constant coefficients depending on λ.

Let's write the impulsive condition (2) and substitute in (15) instead of

$y_-\left(0^-,\lambda\right), y_+\left(0^+,\lambda\right), y_-'\left(0^-,\lambda\right)$ and $y_+'\left(0^+,\lambda\right)$.

We will get the following system of linear equations

$$\begin{cases} A_+ f_2^+(0,\lambda) + B_+ f_2^-(0,\lambda) = \alpha_1\left(A_- f_1^+(0,\lambda) + B_- f_1^-(0,\lambda)\right) + \\ +\alpha_2\left(A_- f_1^{+\prime}(0,\lambda) + B_- f_1^{-\prime}(0,\lambda)\right) \\ A_+ f_2^{+\prime}(0,\lambda) + B_+ f_2^{-\prime}(0,\lambda) = \alpha_3\left(A_- f_1^+(0,\lambda) + B_- f_1^-(0,\lambda)\right) + \\ +\alpha_4\left(A_- f_1^{+\prime}(0,\lambda) + B_- f_1^{-\prime}(0,\lambda)\right) \end{cases} \tag{16}$$

By solving (16) and after making some simplifications, we find:

$$A_+ = \frac{\left[f_1^+(0,\lambda)\left(\alpha_1 f_2^{-\prime}(0,\lambda) - \alpha_3 f_2^-(0,\lambda)\right) + f_1^{+\prime}(0,\lambda)\left(\alpha_2 f_2^{-\prime}(0,\lambda) - \alpha_4 f_2^-(0,\lambda)\right)\right]A_- +}{f_2^+(0,\lambda)\, f_2^{-\prime}(0,\lambda) - f_2^{+\prime}(0,\lambda)\, f_2^-(0,\lambda)}$$
$$\frac{+\left[f_1^-(0,\lambda)\left(\alpha_1 f_2^{-\prime}(0,\lambda) - \alpha_3 f_2^-(0,\lambda)\right) + f_1^{-\prime}(0,\lambda)\left(\alpha_2 f_2^{-\prime}(0,\lambda) - \alpha_4 f_2^-(0,\lambda)\right)\right]B_-}{f_2^+(0,\lambda)\, f_2^{-\prime}(0,\lambda) - f_2^{+\prime}(0,\lambda)\, f_2^-(0,\lambda)} \tag{17}$$

$$B_+ = \frac{\left[f_1^+(0,\lambda)\left(\alpha_1 f_2^{+\prime}(0,\lambda) - \alpha_3 f_2^+(0,\lambda)\right) + f_1^{+\prime}(0,\lambda)\left(\alpha_2 f_2^{+\prime}(0,\lambda) - \alpha_4 f_2^+(0,\lambda)\right)\right]A_- +}{f_2^-(0,\lambda)\, f_2^{+\prime}(0,\lambda) - f_2^{-\prime}(0,\lambda)\, f_2^+(0,\lambda)}$$
$$\frac{+\left[f_1^-(0,\lambda)\left(\alpha_1 f_2^{+\prime}(0,\lambda) - \alpha_3 f_2^+(0,\lambda)\right) + f_1^{-\prime}(0,\lambda)\left(\alpha_2 f_2^{+\prime}(0,\lambda) - \alpha_4 f_2^+(0,\lambda)\right)\right]B_-}{f_2^-(0,\lambda)\, f_2^{+\prime}(0,\lambda) - f_2^{-\prime}(0,\lambda)\, f_2^+(0,\lambda)} \tag{18}$$

If we write (17) – (18) in matrix form, we will get the following:

$$\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \frac{1}{-2i\beta\lambda}\begin{bmatrix} f_2^{-\prime}(0,\lambda) & -f_2^-(0,\lambda) \\ -f_2^{+\prime}(0,\lambda) & f_2^+(0,\lambda) \end{bmatrix}\begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}\begin{bmatrix} f_1^+(0,\lambda) & f_1^-(0,\lambda) \\ f_1^{+\prime}(0,\lambda) & f_1^{-\prime}(0,\lambda) \end{bmatrix}\begin{bmatrix} A_- \\ B_- \end{bmatrix}$$

Then from the impulsive condition (3) we have transfer matrix M satisfying

$$\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = M\begin{bmatrix} A_- \\ B_- \end{bmatrix} \tag{19}$$

where

$$M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = N^{-1}BD \tag{20}$$

with

$$D := \begin{bmatrix} f_1^+(0,\lambda) & f_1^-(0,\lambda) \\ f_1^{+\prime}(0,\lambda) & f_1^{-\prime}(0,\lambda) \end{bmatrix} \tag{21}$$

and

$$N := \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{22}$$

where

$$N_{11} = \left(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}}{\Lambda_n + 2\lambda}\right)$$

$$N_{12} = \left(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}}{\Lambda_n - 2\lambda}\right)$$

$$N_{21} = i\lambda\left(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}}{\Lambda_n + 2\lambda}\right) + \left(\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{i\Lambda_\alpha V_{n\alpha}}{\Lambda_n + 2\lambda}\right)$$

$$N_{22} = -i\lambda\left(1 + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}}{\Lambda_n - 2\lambda}\right) + \left(\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{i\Lambda_\alpha V_{n\alpha}}{\Lambda_n - 2\lambda}\right)$$

It can be shown that $\det N = -2i\beta\lambda$, and $\det N^{-1} = \dfrac{1}{-2i\beta\lambda}$. Then we easily obtain the following:

$$M_{22}(\lambda) = \frac{i}{2\beta\lambda}\Big\{-f_2^{+\prime}(0,\lambda)\Big[\alpha_1 f_1^{-}(0,\lambda) + \alpha_2 f_1^{-\prime}(0,\lambda)\Big] + $$
$$+ f_2^{+}(0,\lambda)\Big[\alpha_3 f_1^{-}(0,\lambda) + \alpha_4 f_1^{-\prime}(0,\lambda)\Big]\Big\}, \tag{23}$$

$$M_{12}(\lambda) = \frac{i}{2\beta\lambda}\Big\{f_2^{-\prime}(0,\lambda)\Big[\alpha_1 f_1^{-}(0,\lambda) + \alpha_2 f_1^{-\prime}(0,\lambda)\Big] - $$
$$- f_2^{-}(0,\lambda)\Big[\alpha_3 f_1^{-}(0,\lambda) + \alpha_4 f_1^{-\prime}(0,\lambda)\Big]\Big\} \tag{24}$$

$$M_{21}(\lambda) = \frac{i}{2\beta\lambda}\Big\{-f_2^{+\prime}(0,\lambda)\Big[\alpha_1 f_1^{+}(0,\lambda) + \alpha_2 f_1^{+\prime}(0,\lambda)\Big] + $$
$$+ f_2^{+}(0,\lambda)\Big[\alpha_3 f_1^{+}(0,\lambda) + \alpha_4 f_1^{+\prime}(0,\lambda)\Big]\Big\} \tag{25}$$

$$M_{11}(\lambda) = \frac{i}{2\beta\lambda}\Big\{f_2^{-\prime}(0,\lambda)\Big[\alpha_1 f_1^{+}(0,\lambda) + \alpha_2 f_1^{+\prime}(0,\lambda)\Big] - $$
$$- f_2^{-}(0,\lambda)\Big[\alpha_3 f_1^{+}(0,\lambda) + \alpha_4 f_1^{+\prime}(0,\lambda)\Big]\Big\} \tag{26}$$

3. Construction of Green's function

Let us consider the non – homogeneous differential equation

$$-y''+q(x)y=\lambda^2\rho(x)y-f(x),\ \ x\in(-\infty,0)\cup(0,\infty) \qquad (27)$$

together with the conditions (2) – (6).

We can represent the general solution of homogeneous differential equation corresponding to equation (27) in the form

$$U(x,\lambda)=\begin{cases} C_1 f_1^+(x,\lambda)+D_1 f_1^-(x,\lambda) & \text{for } -\infty<x<0 \\ C_2 f_1^+(x,\lambda)+D_2 f_1^-(x,\lambda) & \text{for } 0<x<\infty \end{cases}$$

where C_1, D_1, C_2 and D_2 are arbitrary constants.

By applying the standard method of variation of the parameters we will search the general solution of the non – homogeneous linear differential equation (27) in the form

$$U(x,\lambda)=\begin{cases} C_1(x,\lambda) f_1^+(x,\lambda)+D_1(x,\lambda) f_1^-(x,\lambda) & \text{for } -\infty<x<0 \\ C_2(x,\lambda) f_1^+(x,\lambda)+D_2(x,\lambda) f_1^-(x,\lambda) & \text{for } 0<x<\infty \end{cases} \qquad (28)$$

where the functions $C_1(x,\lambda),\ D_1(x,\lambda)$ and $C_2(x,\lambda),\ D_2(x,\lambda)$ satisfies the linear system of equation

$$\begin{cases} C_1'(x,\lambda) f_1^+(x,\lambda)+D_1'(x,\lambda) f_1^-(x,\lambda)=0 \\ C_1'(x,\lambda) f_1^{+\prime}(x,\lambda)+D_1'(x,\lambda) f_1^{-\prime}(x,\lambda)=f(x) \end{cases} \qquad (29)$$

for $x\in(-\infty,0)$ and

$$\begin{cases} C_2'(x,\lambda) f_2^+(x,\lambda)+D_2'(x,\lambda) f_2^-(x,\lambda)=0 \\ C_2'(x,\lambda) f_2^{+\prime}(x,\lambda)+D_2'(x,\lambda) f_2^{-\prime}(x,\lambda)=f(x) \end{cases} \qquad (30)$$

for $x\in(0,\infty)$ respectively. Since

$$w_1(\lambda)=\left(f_1^+(x,\lambda),\ f_1^-(x,\lambda)\right)=-2i\lambda \quad \text{and} \quad w_2(\lambda)=\left(f_2^+(x,\lambda),\ f_2^-(x,\lambda)\right)=-2i\beta\lambda$$

each of the linear system of equations (29) and (30) has a unique solution. These solutions can be expressed as

$$C_1'(x,\lambda)=-\frac{1}{w_1(\lambda)} f_1^-(x,\lambda) f(x) \qquad (31)$$

$$D_1'(x,\lambda)=\frac{1}{w_1(\lambda)} f_1^+(x,\lambda) f(x) \qquad (32)$$

for $x\in(-\infty,0)$ and

$$C_2'(x,\lambda)=-\frac{1}{w_2(\lambda)} f_2^-(x,\lambda) f(x) \qquad (33)$$

$$D_2'(x,\lambda)=\frac{1}{w_2(\lambda)}f_2^+(x,\lambda)f(x) \tag{34}$$

for $x\in(0,\infty)$, respectively. From equations (31) – (34), the following relations are obtained:

$$C_1(x,\lambda)=-\frac{1}{w_1(\lambda)}\int_{-\infty}^{x}f_1^-(t,\lambda)f(t)\,dt+C_1 \qquad x\in(-\infty,0)$$

$$D_1(x,\lambda)=-\frac{1}{w_1(\lambda)}\int_{x}^{0}f_1^+(t,\lambda)f(t)\,dt+D_1 \qquad x\in(-\infty,0)$$

$$C_2(x,\lambda)=-\frac{1}{w_2(\lambda)}\int_{0}^{x}f_2^-(t,\lambda)f(t)\,dt+C_2 \qquad x\in(0,\infty)$$

$$D_2(x,\lambda)=-\frac{1}{w_2(\lambda)}\int_{\infty}^{x}f_2^+(t,\lambda)f(t)\,dt+D_2 \qquad x\in(0,\infty)$$

where C_1,D_1,C_2 and D_2 are arbitrary constants. Substituting the above equations in (28), the general solution of non – homogeneous linear differential equation (27) are obtained as

$$U_-(x,\lambda)=-\frac{f_1^+(x,\lambda)}{w_1(\lambda)}\int_{-\infty}^{x}f_1^-(t,\lambda)f(t)\,dt+C_1f_1^+(x,\lambda)-\frac{f_1^-(x,\lambda)}{w_1(\lambda)}\int_{x}^{0}f_1^+(t,\lambda)f(t)\,dt+$$
$$+D_1f_1^-(x,\lambda)$$

for $-\infty<x<0$ (35)

$$U_+(x,\lambda)=-\frac{f_2^+(x,\lambda)}{w_2(\lambda)}\int_{0}^{x}f_2^-(t,\lambda)f(t)\,dt+C_2f_2^+(x,\lambda)-\frac{f_2^-(x,\lambda)}{w_2(\lambda)}\int_{x}^{\infty}f_2^+(t,\lambda)f(t)\,dt+$$
$$+D_2f_2^-(x,\lambda)$$

for $0<x<\infty$. (36)

Since $U_-(x,\lambda)\in L_2(-\infty,0)$ and $U_+(x,\lambda)\in L_2(0,\infty)$, $C_1=0$ and $D_2=0$.

Now by using the impulsive condition (2), let's find C_2 and D_1:

$$-\frac{f_2^-(0,\lambda)}{w_2(\lambda)}\int_{0}^{\infty}f_2^+(t,\lambda)f(t)\,dt+C_2f_2^+(0,\lambda)=\alpha_1\left[-\frac{f_1^+(0,\lambda)}{w_1(\lambda)}\int_{-\infty}^{0}f_1^-(t,\lambda)f(t)\,dt+D_1f_1^-(0,\lambda)\right]+$$
$$+\alpha_2\left[-\frac{f_1^{+\prime}(0,\lambda)}{w_1(\lambda)}\int_{-\infty}^{0}f_1^-(t,\lambda)f(t)\,dt+D_1f_1^{-\prime}(0,\lambda)\right]$$

$$-\frac{f_2^{-\prime}(0,\lambda)}{w_2(\lambda)}\int_{0}^{\infty}f_2^+(t,\lambda)f(t)\,dt+C_2f_2^{+\prime}(0,\lambda)=\alpha_3\left[-\frac{f_1^+(0,\lambda)}{w_1(\lambda)}\int_{-\infty}^{0}f_1^-(t,\lambda)f(t)\,dt+D_1f_1^-(0,\lambda)\right]+$$
$$+\alpha_4\left[-\frac{f_1^{+\prime}(0,\lambda)}{w_1(\lambda)}\int_{-\infty}^{0}f_1^-(t,\lambda)f(t)\,dt+D_1f_1^{-\prime}(0,\lambda)\right]$$

By solving the system of equations above, we obtain the following result:

$$D_1 = \frac{M_{21}(\lambda)\int\limits_{-\infty}^{0} f_1^-(t,\lambda)\, f(t)\,dt - \frac{i}{2\beta\lambda}\int\limits_{0}^{\infty} f_2^+(t,\lambda)\, f(t)\,dt}{M_{22}(\lambda)}$$

$$C_2 = \frac{-M_{12}(\lambda)\int\limits_{0}^{\infty} f_2^+(t,\lambda)\, f(t)\,dt - \frac{i}{2\beta\lambda}\int\limits_{-\infty}^{0} f_1^-(t,\lambda)\, f(t)\,dt}{M_{22}(\lambda)}$$

Finally, by substituting the coefficients C_i and D_i $(i=1,2)$ in (35) and (36), the following formula is obtained for the resolvent $U(x,\lambda)$:

Furthermore, by using the representations

$$f^+(x,\lambda) = \begin{cases} f_1^+(x,\lambda) & \text{for } x\in(-\infty,0) \\ f_2^+(x,\lambda) & \text{for } x\in(0,\infty) \end{cases}$$

$$f^-(x,\lambda) = \begin{cases} f_1^-(x,\lambda) & \text{for } x\in(-\infty,0) \\ f_2^-(x,\lambda) & \text{for } x\in(0,\infty) \end{cases}$$

this formula can be rewritten in the form

$$U(x,\lambda) = -\frac{i\, f^+(x,\lambda)}{2\beta\lambda M_{22}(\lambda)}\int\limits_{-\infty}^{x} f^-(t,\lambda) f(t)\,dt - \frac{i\, f^-(x,\lambda)}{2\beta\lambda M_{22}(\lambda)}\int\limits_{x}^{\infty} f^+(t,\lambda) f(t)\,dt +$$

$$+\begin{cases} \frac{M_{21}(\lambda)}{M_{22}(\lambda)} f_1^-(x,\lambda)\int\limits_{-\infty}^{0} f_1^-(t,\lambda)\, f(t)\,dt & x\in(-\infty,0) \\ -\frac{M_{12}(\lambda)}{M_{22}(\lambda)} f_2^+(x,\lambda)\int\limits_{0}^{\infty} f_2^+(t,\lambda)\, f(t)\,dt & x\in(0,\infty) \end{cases} \tag{37}$$

$$U(x,\lambda) = \begin{cases} -\frac{f_1^+(x,\lambda)}{w_1(\lambda)}\int\limits_{-\infty}^{x} f_1^-(t,\lambda) f(t)\,dt - \frac{f_1^-(x,\lambda)}{w_1(\lambda)}\int\limits_{x}^{0} f_1^+(t,\lambda) f(t)\,dt + \\ +\frac{M_{21}(\lambda)}{M_{22}(\lambda)} f_1^-(x,\lambda)\int\limits_{-\infty}^{0} f_1^-(t,\lambda)\, f(t)\,dt - \frac{f_1^-(x,\lambda)}{M_{22}(\lambda)}\frac{i}{2\beta\lambda}\int\limits_{0}^{\infty} f_2^+(t,\lambda) f(t)\,dt & x\in(-\infty,0) \\ -\frac{f_2^+(x,\lambda)}{w_2(\lambda)}\int\limits_{0}^{x} f_2^-(t,\lambda)\, f(t)\,dt - \frac{f_2^-(x,\lambda)}{w_2(\lambda)}\int\limits_{x}^{\infty} f_2^+(t,\lambda)\, f(t)\,dt \\ -\frac{M_{12}(\lambda)}{M_{22}(\lambda)} f_2^+(x,\lambda)\int\limits_{0}^{\infty} f_2^+(t,\lambda)\, f(t)\,dt - \frac{f_2^+(x,\lambda)}{M_{22}(\lambda)}\frac{i}{2\beta\lambda}\int\limits_{-\infty}^{0} f_1^-(t,\lambda)\, f(t)\,dt & x\in(0,\infty) \end{cases}$$

Thus, the resolvent of the boundary-value impulsive problem is obtained. We can find Green's function from the resolvent (37) easily. Namely, denoting

$$G(x,t,\lambda) = \begin{cases} -\frac{i}{2\beta\lambda M_{22}(\lambda)} f^+(x,\lambda)\, f^-(t,\lambda) & t<x, \quad x\neq 0, t\neq 0 \\ -\frac{i}{2\beta\lambda M_{22}(\lambda)} f^-(x,\lambda)\, f^+(t,\lambda) & x<t, \quad x\neq 0, t\neq 0 \end{cases}$$

We can rewrite the resolvent (37) in the next form:

$$U(x,\lambda)=\int_{-\infty}^{\infty} G(x,t,\lambda)f(t)\,dt+\begin{cases}\dfrac{M_{21}(\lambda)}{M_{22}(\lambda)} f_1^-(x,\lambda)\displaystyle\int_{-\infty}^{0} f_1^-(t,\lambda)\,f(t)\,dt & x\in(-\infty,0)\\ -\dfrac{M_{12}(\lambda)}{M_{22}(\lambda)} f_2^+(x,\lambda)\displaystyle\int_{0}^{\infty} f_2^+(t,\lambda)\,f(t)\,dt & x\in(0,\infty)\end{cases}$$

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