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$MAP/PH/2$ DOUBLE-ORBIT RETRIAL INVENTORY SYSTEM WITH WORKING VACATION AND RE-SERVICE

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ABSTRACT

The current study focuses on the double orbit retrial inventory queueing reliable and unreliable servers. The customer arrives using a Markovian arr and the service time is allocated based on the phase-type distribution. If prim cannot access the server at first, they enter an infinite orbit. After the service the customer will have the option to join the finite orbit for another service system because they were satisfied. If a customer finds that the finite orbit needs re-service, they are considered lost. If there are no customers on the syst service is finished, server 1 proceeds with the close-down procedure. The se vacation after closing down. Server-1 is idle unless a customer arrives on vac served slowly. Re-service is the only thing server 2 offers. If server-1 fails c periods, the current customer is frozen until repair, after which servi Inventory is replenished exponentially under an (s, S) policy. Using the mat approach, steady-state analysis is performed, and various performance evaluated numerically and graphically.

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1. INTRODUCTION

Retrial queueing systems, also known as queues with repeated attempts, are appropriate mathematical representations of modern contact centers, wireless networks, etc., because they account for the important factor of customer behavior. The customer will make random attempts to obtain access if the required resource is unavailable when they arrive. Retry queues have been the subject of plenty of scholarly work, and the relevant literature is included in the survey articles by [1]. In their study, [2] examined a conventional retrial queueing inventory system with a two-component demand rate. When two orbit queues with infinite capacity are assumed, the joint stationary orbit queue length distribution can be used to calculate the probability using the theory of Riemann (-Hilbert) problems introduced by [3]. In [4], retrial queues with finite capacity and geometric loss was examined. According to [5], the literature on the algorithmic and computational aspects of retrial queues was reviewed.

The double-orbit finite retrial queues with priority clients and service disruptions have been studied by [6]. In [7], the author has examined the optimal approach for a double orbit retrial queue with imperfect service and vacation interruption. Wang [8] conducted research on a retrial make-to-stock system based on a double-ended queue. While the queue length was positive, customers were waiting in the retrial queue; when it was negative, the inventory system contained only products. In order to calculate stationary distributions, Avrachenkov et al. [9], the authors examined a single-server retrial system that was fed by two customer types, each of which entered its own infinite-capacity orbit. They suggested approximate algorithms through orbit truncation and treated the underlying Markov chain as an asymptotically quasi-Toeplitz Markov chain (AQTMC).

Feedback is another often occurring event in queuing systems. Queues with feedback (QFB) are those where each consumer that is served either enters the system with a certain probability or departs forever with a complimentary probability. An $M/M/1$ retrial-queueing system with feedback and server breakdowns is presented by [10]. The matrix-geometric method is used to determine performance measures, including graphical representations, and feedback happens when dissatisfied clients return to the orbit for another service attempt. [11] conducted research on the retrial queueing model with re-service. According to [12], a queueing inventory system with consumer feedback and returns of purchased items was stated. Ayyappan and Archana [13] concentrate on two-server tandem queues with working vacation, feedback, re-service, unreliable servers, interrupted shutdowns, and impatient clients.

When the service is finished and the server discovers that there are no customers with positive inventory in the system, the server takes a vacation. The server may provide a lower service rate than usual if customers arrive while on vacation. Servi and Finn [14] established the concept of working vacation. In [15], the author examined a single-server working vacation queuing system with interconnected arrival and service operations. Note that in [16], the authors discussed the MAP/Ek/1 Queue with Working Vacation. Ayyappan and Arulmozhi [17] examine the retrial queueing inventory system with orbital client collision, a constant retrial rate, working vacation, flush out, balking, breakdown, and repair.

According to [18], they examined a queuing system in which two different service devices served a single demand at varying service rates and energy unit consumption. Stability of queueing systems with impatience, balking and non-persistence of customers was analyzed by [19]. Customers may balk when the server is on vacation or experiencing a breakdown, as assumed by [20]. A queueing inventory system with impatient customers and working vacations was studied by [21].

The literature on stochastic inventory systems with single and multiple commodities is quite large. We provide some of the significant research conducted in two commodity stochastic inventory systems in this part. [22] and [23] looked at problems with inventory of two commodities with Markov change in demands. Note that in [24], the authors studied a single-server queueing system and two-commodity inventory where the buffer capacities for both commodity stocks are finite and customer demands (for either or both commodities) have been represented by given probabilities. In [25], a two-commodity queueing inventory system with individual ordering policy, positive service time, positive lead time, and random common lifetimes for each commodity was examined.

This is how the paper is organized. A thorough description of the model can be found in 2. Section 3 explains matrix formation and a number of notations. The invariant probability vector, the rate matrix R and the stability condition are all obtained in section 4. The analysis of the busy period is described in Section 5. The performance measurements are listed in 6.

Section 7 contains some numerical and graphical results. The conclusion is presented in Section 8.

2 MATHEMATICAL MODEL

This study examines a two-server retrial inventory queuing model with two orbits. For primary customers who are unable to reach the server upon arrival, the first orbit is infinite, whereas the second orbit is finite for re-service customers. The following is an explanation of the model's basic operation:

- Arrival process: The customer arrives in accordance with MAP with parameter matrices (D_0, D_1) of order m. Let D_0 be the transition matrix that indicates that no customers have arrived, and let D_1 be the transition rate matrix that indicates that customers have arrived. Here, we assumed that if a customer arrives and discovers that server-1 is idle with positive inventory, they will receive service from server-1, which serves as their primary service provider; if not, they will join orbit I (infinite orbit). Following primary service, the customer either leaves the system because they are satisfied with the service or moves into orbit II (finite capacity) for another service.
- Service Process: The two servers provide the service with positive inventory at two distinct service rates. The predicted service rates for servers 1 and 2 are $\mu_1 = [\gamma_1(-U_1)^{-1}e_{n_1}]$ and $\mu_2 = [\gamma_2(-U_2)^{-1}e_{n_2}]$ respectively. In this case, server- 2 is always accessible for re-service, while server-1 is operating in normal mode, vacation mode, shutdown, breakdown, and repair. Each server will receive one item from the server upon completion of the service. The customer either enters an orbit (finite) for reservice after server- 1 completes the primary service or leaves the system entirely since they were satisfied. The service time of both servers follows phase-type distribution with representation (γ_1, U_1) of order n_1, such that $U_1^0 + U_1 e = 0$ and (γ_2, U_2) of order n_2, such that $U_2^0 + U_2 e = 0$, respectively.
- Breakdown and repair state: The server- 1 is prone to breaking down during busy periods and is immediately repaired. Server-1 pauses the customer who is currently in service, putting them in a frozen condition. Once the repair is finished, server- 1 provides the interrupted consumer with new service. For server-2, there is no malfunction. ψ and δ represent the exponential distribution of breakdown and repair time, respectively.
- Retrial process: Every customer in the orbit makes an independent effort to regain access. The attempt is successful and the customer takes up a free server right away. It is assumed that the customers' retrial time is exponential, with parameters σ_2 (finite orbit) and σ_1 (infinite orbit), respectively.
- Re-service period: Following primary service from server-1, a customer who is satisfied with the service may either leave the system with probability f_1 or enter orbit-II for re-service from server- 2 with probability f_2.
- Close down: If there are no customers in the system once the service is finished, server-1 proceeds to close down. Close down time is represented by the symbol ϕ and is assumed to have an exponential distribution.
- Vacation time: When the close-down procedure is finished and there are no customers in the system, server-1 starts the vacation period. Its rate is represented by η, which is followed by an exponential distribution. When a customer arrives with positive inventory during vacation time, server-1 serves them at a slow rate in phase-type

distribution, represented as $(\alpha, \theta U_1)$ of order n, with a working vacation rate of $\theta\mu_1$. If server-1 gets an empty system at the time of working vacation mode, then server-1 moves to idle in working vacation mode. After vacation completion, the server will return to its normal working mode if there are both positive items and customers in the system. Otherwise, server-1 becomes idle if zero customers in the system or zero items or both.

- Replenishment: When a customer completes a service, one item is removed from stock. We consider a stochastic inventory system with two different items in stock; one item is for server-1 (I commodity) and the other item is for server-2 (II-commodity). The maximum storage capacity for the i^{th} commodity is $S_i(\text{i} = 1,2)$. An (s_1, S_1) ordering policy is adopted for the first commodity, (s_2, S_2) ordering policy is adopted for the second commodity. When the on-hand inventory level of the I-commodity drops to a prefixed level s_1, an order for S_1 units is placed. The lead time of commodity-I is exponentially distributed with parameter β. In addition, assume that an emergency replenishment of one item with zero lead time takes place when the on-hand inventory level decreases to zero. The emergency replenishment is incorporated in the system to ensure customer satisfaction for re-service customers in orbit-II. Refer to figure1 for a schematic representation of the model.
-

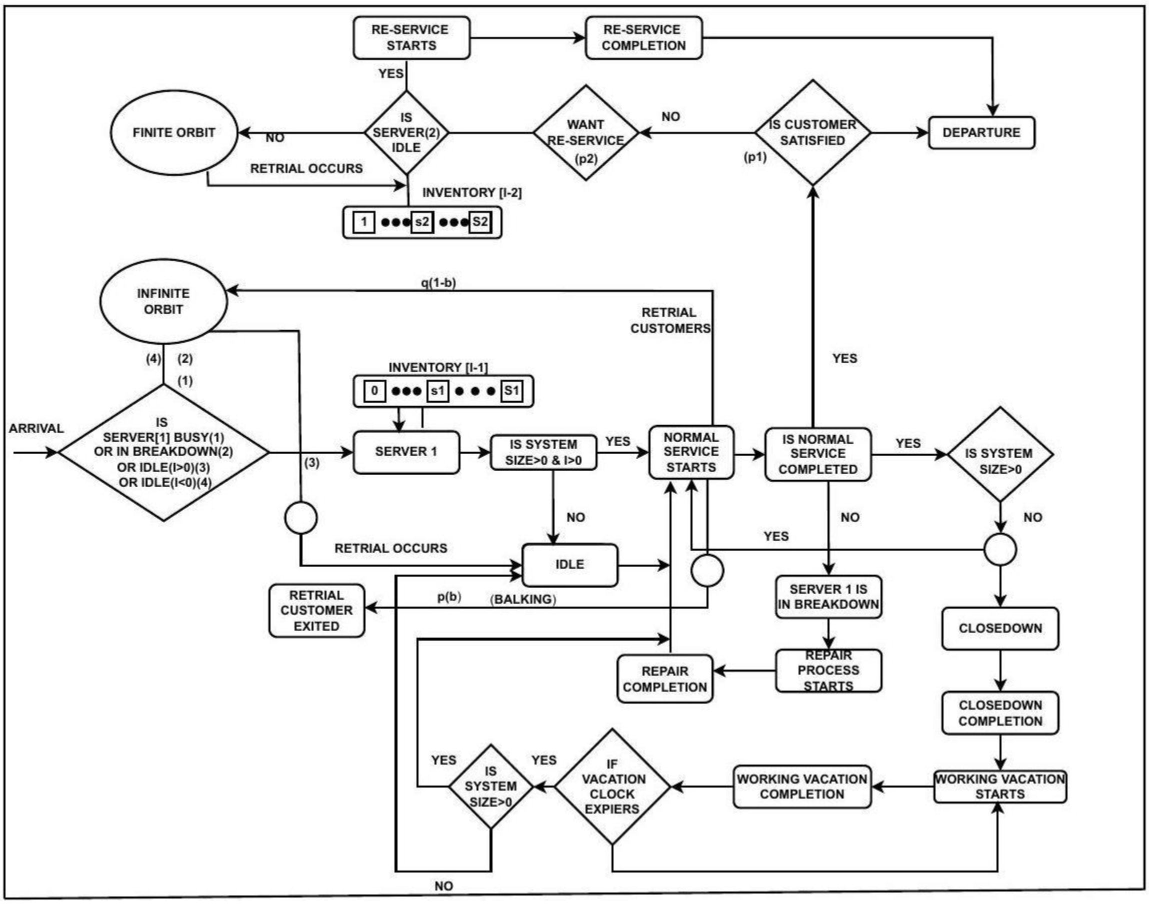

Figure 1: Schematic representation

3 GENERATION MATRIX

Let us describe the few notations of this model that are followed by the Quasi Birth-Death (QBD) process generating matrix as follows:

ABBREVIATION AND NOTATIONS

- CTMC- Continuous Time Markov Chain
- QBD process - Quasi-Birth-Death Process
- QIS- Queueing Inventory System
- MAM - Matrix Analytic Method
- e - entries in a column's vector are one;
- I_n - Identity matrix of order n
- 0 - Matrix whose entries are 0, of appropriate size
- $\otimes$ - Kronecker product of two matrices of different dimensions;
- $\oplus$ - Addition of two matrices of different dimensions;
- $N_1(t)$ indicates the number of customers in an infinite orbit;
- $N_2(t)$ indicates the number of customers in the finite orbit;
- $J_1(t)$ represents the server-1 status;
- $J_2(t)$ represents the server-2 status;
- $I_1(t)$ represents the stock level of commodity-I;
- $I_2(t)$ represents the stock level of commodity-II;
- $S_1(t)$ represents the service phases of server-1;
- $S_2(t)$ represents the service phases of server-2;
- A(t) represents the arrival phases; at time t, where

$$J_1(t) = \begin{cases} 0, & \text{server- 1 is idle in normal mode ,} \\ 1, & \text{server- 1 is busy in normal mode ,} \\ 2, & \text{server- 1 is in repair,} \\ 3, & \text{server- 1 is in closedown,} \\ 4, & \text{server- 1 is idle in working vacation mode,} \\ 5, & \text{server- 1 is busy in working vacation mode} \end{cases}$$

$$J_2(t) = \begin{cases} 0, & \text{server- 2 is idle,} \\ 1, & \text{server- 2 is working with re-service} \end{cases}$$

It is obvious that $\{N_1(t), N_2(t), J_1(t), J_2(t), I_1(t), I_2(t), S_1(t), S_2(t), A(t): t \geq 0\}$ is a CTMC with State Space ia sa follows:

$$\omega = \bigcup_{d_1=0}^{\infty} \chi(d_1)\#(1)$$

$$
\begin{aligned}
\chi(d_1) =&\{(d_1,d_2,0,0,d_5,d_6,d_9): 0 \le d_2 \le M, 0 \le d_5 \le S_1, 1 \le d_6, \le S_2, 1 \le d_9 \le m\} \\
&\cup\{(d_1,d_2,0,1,d_5,d_6,d_8,d_9): 0 \le d_2 \le M, 0 \le d_5 \le S_1, 1 \le d_6, \le S_2, 1 \le d_8 \le n_2 \\
&\quad 1 \le d_9 \le m\} \\
&\cup\{(d_1,d_2,1,0,d_5,d_6,d_7,d_9): 0 \le d_2 \le M, 1 \le d_5 \le S_1, 1 \le d_6 \le S_2 \\
&\quad 1 \le d_7 \le n_1, 1 \le d_9 \le m\} \\
&\cup\{(d_1,d_2,1,1,d_5,d_6,d_7,d_8,d_9): 0 \le d_2 \le M, 1 \le d_5 \le S_1, 1 \le d_6 \le S_2 \\
&\quad 1 \le d_7 \le n_1, 1 \le d_8 \le n_2, 1 \le d_9 \le m\} \\
&\cup\{(d_1,d_2,2,0,d_5,d_6,d_9): 0 \le d_2 \le M, 1 \le d_5 \le S_1, 1 \le d_6 \le S_2, 1 \le d_9 \le m\} \\
&\cup\{(d_1,d_2,2,1,d_5,d_6,d_8,d_9): 0 \le d_2 \le M, 1 \le d_5 \le S_1, 1 \le d_6 \le S_2 \\
&\quad 1 \le d_8 \le n_2, 1 \le d_9 \le m\} \\
&\cup\{(d_1,d_2,3,0,d_5,d_6,d_9): 0 \le d_2 \le M, 0 \le d_5 \le S_1, 1 \le d_6 \le S_2, 1 \le d_9 \le m\} \\
&\cup\{(d_1,d_2,3,1,d_5,d_6,d_8,d_9): 0 \le d_2 \le M, 0 \le d_5 \le S_1, 1 \le d_6 \le S_2, 1 \le d_8 \le n_2 \\
&\quad 1 \le d_9 \le m\} \\
&\cup\{(d_1,d_2,4,0,d_5,d_6,d_9): 0 \le d_2 \le M, 1 \le d_5 \le S_1, 1 \le d_6 \le S_2, 1 \le d_9 \le m\} \\
&\cup\{(d_1,d_2,4,1,d_5,d_6,d_8,d_9): 0 \le d_2 \le M, 1 \le d_5 \le S_1, 1 \le d_6 \le S_2, 1 \le d_8 \le n_2 \\
&\quad 1 \le d_9 \le m\} \\
&\cup\{(d_1,d_2,5,0,d_5,d_6,d_7,d_9): 0 \le d_2 \le M, 1 \le d_5 \le S_1, 1 \le d_6 \le S_2, 1 \le d_7 \le n_1 \\
&\quad 1 \le d_9 \le m\} \\
&\cup\{(d_1,d_2,5,1,d_5,d_6,d_7,d_8,d_9): 0 \le d_2 \le M, 1 \le d_5 \le S_1, 1 \le d_6 \le S_2 \\
&\quad 1 \le d_7 \le n_1, 1 \le d_8 \le n_2, 1 \le d_9 \le m\}
\end{aligned}
$$

QUASI BIRTH AND DEATH (QBD) PROCESS'S CONSTRUCTION

The following is the Markov chain's generator matrix

$$
Q = \begin{bmatrix}
B_0 & A_0 & 0 & 0 & 0 & 0 & \dots \\
A_2 & A_1 & A_0 & 0 & 0 & 0 & \dots \\
0 & A_2 & A_1 & A_0 & 0 & 0 & \dots \\
0 & 0 & A_2 & A_1 & A_0 & 0 & \dots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots
\end{bmatrix}, \#(2)
$$

The following defines each entry in Q's block matrices:

$$
B_0 = \begin{bmatrix}
B_0^{11} & B_0^{12} & B_0^{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B_0^{21} & B_0^{22} & 0 & B_0^{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B_0^{33} & B_0^{34} & 0 & 0 & B_0^{37} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B_0^{43} & B_0^{44} & 0 & B_0^{46} & 0 & B_0^{48} & 0 & 0 & 0 & 0 \\
0 & 0 & B_0^{53} & 0 & B_0^{55} & B_0^{56} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & B_0^{64} & B_0^{65} & B_0^{66} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & B_0^{77} & B_0^{78} & B_0^{79} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & B_0^{87} & B_0^{88} & 0 & B_0^{810} & 0 & 0 \\
B_0^{91} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_0^{99} & B_0^{910} & B_0^{911} & 0 \\
0 & B_0^{102} & 0 & 0 & 0 & 0 & 0 & 0 & B_0^{109} & B_0^{1010} & 0 & 0 \\
0 & 0 & B_0^{113} & 0 & 0 & 0 & 0 & 0 & B_0^{119} & 0 & B_0^{1111} & B_0^{1112} \\
0 & 0 & 0 & B_0^{124} & 0 & 0 & 0 & 0 & 0 & B_0^{1210} & B_0^{1211} & B_0^{1212}
\end{bmatrix},
$$

$$
B_0^{11} = \begin{bmatrix} K_1 & 0 \\ 0 & I_{S_1} \otimes K_2 \end{bmatrix}, B_0^{13} = I_{S_1+1} \otimes K_4,
$$

$$
K_2 = \begin{bmatrix}
N_4 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_5 \\
\mathbf{0} & N_4 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_5 \\
\mathbf{0} & \mathbf{0} & N_4 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_5 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_4 & \mathbf{0} & \dots & \mathbf{0} & N_5 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_6 & \dots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_6 & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_6
\end{bmatrix}, K_1 = \begin{bmatrix}
N_1 & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_2 \\
\mathbf{0} & N_1 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_2 \\
\mathbf{0} & \mathbf{0} & N_1 & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_1 & \mathbf{0} & \dots & \mathbf{0} & N_2 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_3 & \dots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\mathbf{0} & \mathbf{0} & \mathrm{b}f0 & \dots & \mathbf{0} & \mathbf{0} & \dots & N_3 & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_3
\end{bmatrix},
$$

where $N_1 = I_{s_1+1} \otimes I_{S_2} \otimes D_0 - \beta I_m, N_2 = e_{s_1+1} \otimes I_{S_2} \otimes \beta I_m,$

$$N_3 = I_{(S_1+1)} - (s_1+1) \otimes I_{S_2} \otimes D_0, N_4 = I_{s_1+1} \otimes I_{S_2} \otimes D_0 - (\beta+\sigma_2) I_m,$$

$$N_2 = N_5, N_6 = I_{(S_1+1)} - (s_1+1) \otimes I_{S_2} \otimes D_0 - \sigma_2,$$

$$B_0^{12} = \begin{bmatrix} 0 & 0 \\ I_{S_1} \otimes K_3 & 0 \end{bmatrix}, K_3 = I_{S_1+1} \otimes N_7, N_7 = I_{S_1+1} \otimes I_{S_2} \otimes \gamma_2 \otimes \sigma_2 I_m,$$

$$K_4 = \begin{bmatrix} 0 \\ I_{S_1+1} \otimes N_8 \end{bmatrix}, B_0^{21} = I_{S_1+1} \otimes K_5, N_8 = I_{S_2} \otimes \gamma_1 \otimes D_1,$$

$$K_5 = \begin{bmatrix} N_9 & 0 \\ I_{S_1+1} \otimes & 0 \end{bmatrix}, B_0^{22} = I_{S_1+1} \otimes K_6, N_9 = U2 \otimes I_m,$$

$$K_6 = \begin{bmatrix} N_{10} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{11} \\ \mathbf{0} & N_{10} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{11} \\ \mathbf{0} & \mathbf{0} & N_{10} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{11} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{10} & \mathbf{0} & \dots & \mathbf{0} & N_{11} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{12} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{12} \end{bmatrix}, B_0^{24} = I_{S_1+1} \otimes K_7,$$

where $N_{10} = I_{S_2} \otimes (U2 \oplus D_0) - \beta I_{n_2 m}, N_{11} = e_{S_2} \otimes I_{n_2} \otimes \beta I_m \ N_{12} = I_{S_2} \otimes (U2 \oplus D_0),$

$$K_7 = \begin{bmatrix} 0 \\ I_{S_1} \otimes N_{13} \end{bmatrix}, B_0^{33} = \begin{bmatrix} K_8 & 0 \\ 0 & I_{S_1-1} \otimes K_9 \end{bmatrix}, N_{13} = I_{S_2} \otimes I_{n_2} \otimes \gamma_1 \otimes D_0$$

$$K_8 = \begin{bmatrix} N_{14} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{15} \\ \mathbf{0} & N_{14} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{15} \\ \mathbf{0} & \mathbf{0} & N_{14} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{15} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{14} & \mathbf{0} & \dots & \mathbf{0} & N_{15} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{16} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{16} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{16} \end{bmatrix}, K_9 = \begin{bmatrix} N_{17} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{18} \\ \mathbf{0} & N_{17} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{18} \\ \mathbf{0} & \mathbf{0} & N_{17} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{18} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{17} & \mathbf{0} & \dots & \mathbf{0} & N_{18} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{19} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{19} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{19} \end{bmatrix},$$

where $N_{14} = I_{s_1} \otimes I_{S_2} \otimes (U1 \oplus D_0) - \beta I_{n_1 m}, N_{15} = N_{18} = e_{s_1} \otimes I_{S_2} \otimes \beta I_{n_1} \otimes \beta I_m$

$N_{16} = I_{(S_1-s_1)} \otimes I_{S_2} \otimes (U1 \oplus D_0), N_{17} = I_{s_1} \otimes I_{S_2} \otimes (U1 \oplus D_0) - (\beta+\sigma_2) I_{n_1 m},$

$N_{19} = I_{(S_1-s_1)} \otimes I_{S_2} \otimes (U1 \oplus D_0) - \sigma_2 I_{n_1 m},$

$$B_0^{34} = \begin{bmatrix} 0 \\ I_{S_1-1} \otimes K_{10} & 0 \end{bmatrix}, K_{10} = \begin{bmatrix} N_{20} & 0 \\ 0 & I_{S_1} \otimes N_{20} \end{bmatrix}, N_{20} = I_{S_1 S_2} \otimes I_{n_1} \otimes \gamma_2 \otimes \sigma_2 I_m,$$

$$B_0^{37} = \begin{bmatrix} K_{11} & K_{12} & 0 & \dots & 0 & 0 \\ 0 & K_{11} & K_{12} & \dots & 0 & 0 \\ 0 & 0 & K_{11} & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & K_{11} & K_{12} \\ 0 & 0 & 0 & \dots & 0 & K_{11}+K_{12} \end{bmatrix},$$

where $K_{11} = [I_{S_1 S_2} \otimes f_1 U1^0 \otimes I_m \quad 0], K_{12} = [I_{S_1 S_2} \otimes f_2 U1^0 \otimes I_m \quad 0],$

$B_0^{43} = [I_{M+1} \otimes I_{S_1} \otimes K_{13}],$

$$K_{13} = \begin{bmatrix} N_{21} & 0 \\ I_{S_1-1} \otimes N_{21} & 0 \end{bmatrix}, B_0^{44} = I_{S_1} \otimes K_{14},$$

where $N_{21} = I_{n_1} \otimes U2^0 \otimes I_m, B_0^{46} = I_{M+1} \otimes I_{S_1} \otimes I_{S_2} \otimes e_{n_1} \otimes \tau I_{n_2 m}$

$$K_{14} = \begin{bmatrix} N_{22} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{23} \\ \mathbf{0} & N_{22} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{23} \\ \mathbf{0} & \mathbf{0} & N_{22} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{23} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{22} & \mathbf{0} & \dots & \mathbf{0} & N_{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{24} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{24} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{24} \end{bmatrix}, B_0^{48} = \begin{bmatrix} K_{17} & K_{18} & 0 & \dots & 0 & 0 \\ 0 & K_{17} & K_{18} & \dots & 0 & 0 \\ 0 & 0 & K_{12} & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & K_{17} & K_{18} \\ 0 & 0 & 0 & \dots & 0 & K_{17}+K_{18} \end{bmatrix},$$

where $N_{22} = I_{s_1} \otimes I_{S_2} \otimes (U2 \oplus D_0) - (\tau+\beta) I_{n_1 n_2 m}, N_{23} = e_{s_1} \otimes I_{S_2} \otimes I_{n_1 n_2} \otimes \beta I_m$

$N_{24} = I_{S_1-s_1} \otimes I_{S_2} \otimes [(U1 \oplus U2) \oplus D_0] - \tau I_{n_1 n_2 m}, K_{15} = [I_{S_1} \otimes I_{S_2} \otimes f_1 U1^0 \otimes I_{n_2} \otimes I_m \ 0],$

$K_{16} = [I_{S_1} \otimes I_{S_2} \otimes f_2 U1^0 \otimes I_{n_2} \otimes I_m \ 0], B_0^{53} = [I_{M+1} \otimes I_{S_1} \otimes I_{S_2} \otimes \gamma_1 \otimes \delta I_m],$

$$B_0^{55} = \begin{bmatrix} K_{17} & 0 \\ 0 & I_{S_1-1} \otimes K_{18} \end{bmatrix}, K_{17} = \begin{bmatrix} N_{25} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{26} \\ \mathbf{0} & N_{25} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{26} \\ \mathbf{0} & \mathbf{0} & N_{25} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{26} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{25} & \mathbf{0} & \dots & \mathbf{0} & N_{26} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{27} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{27} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{27} \end{bmatrix},$$

$N_{25} = I_{s_1} \otimes I_{S_2} \otimes D_0 - (\delta+\beta) I_m, N_{26} = e_{s_1} \otimes I_{S_2} \otimes \beta I_m \ N_{27} = I_{S_1-s_1} \otimes I_{S_2} \otimes D_0 - \delta I_m,$

$$K_{18}=\begin{bmatrix} N_{28} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{29}\\ \mathbf{0} & N_{28} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{29}\\ \mathbf{0} & \mathbf{0} & N_{28} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{29}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{28} & \mathbf{0} & \dots & \mathbf{0} & N_{29}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{30} & \dots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{30} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{30} \end{bmatrix}, B_0^{56}=\begin{bmatrix} 0 & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & 0\\ K_{19} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & 0\\ \mathbf{0} & K_{19} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \dots & K_{19} & \mathbf{0} & \dots & \mathbf{0} & 0\\ \mathbf{0} & \mathbf{0} & \dots & 0 & K_{19} & \dots & 0 & \mathbf{0}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \dots & K_{19} & 0 \end{bmatrix},$$

where $N_{28}=I_{s_1}\otimes I_{S_2}\otimes D_0-(\delta+\beta+\sigma_2)I_m, N_{29}=e_{s_1}\otimes I_{S_2}\otimes \beta I_m$

$N_{30}=I_{S_1-s_1}\otimes I_{S_2}\otimes D_0-(\delta+\sigma_2)I_m, K_{19}=I_{S_1S_2}\otimes\gamma_2\otimes\sigma_2 I_m,$

$B_0^{64}=I_{M+1}\otimes I_{S_1S_2}\otimes\gamma_1\otimes I_{n_2}\otimes\delta I_m,$

$B_0^{65}=[I_{S_1}\otimes K_{20}], B_0^{66}=[I_{S_1}\otimes K_{21}], K_{20}=I_{S_1}\otimes I_{S_2}\otimes U2^0\otimes I_m,$

$$K_{21}=\begin{bmatrix} N_{31} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{32}\\ \mathbf{0} & N_{31} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{32}\\ \mathbf{0} & \mathbf{0} & N_{31} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{32}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{31} & \mathbf{0} & \dots & \mathbf{0} & N_{32}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{33} & \dots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{33} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{33} \end{bmatrix}, B_0^{77}=\begin{bmatrix} K_{22} & 0\\ 0 & I_{S_1}\otimes K_{23}\end{bmatrix},$$

where $N_{31}=I_{s_1}\otimes I_{S_2}\otimes(U2+D_0)-(\delta+\beta)I_{n_2m}, N_{32}=e_{s_1}\otimes I_{S_2}\otimes I_{n_2}\otimes\beta I_m$

$N_{33}=I_{S_1-s_1}\otimes I_{S_2}\otimes(U2+D_0)-\delta I_{n_2m},$

where $N_{34}=I_{s_1+1}\otimes I_{S_2}\otimes D_0-(\delta+\beta)I_m, N_{35}=e_{s_1+1}\otimes I_{S_2}\otimes\beta I_m$

$N_{36}=I_{(S_1+1)-(s_1+1)}\otimes I_{S_2}\otimes D_0-\phi I_m, N_{37}=I_{s_1+1}\otimes I_{S_2}\otimes D_0-(\phi+\sigma_2+\beta)I_m$

$N_{38}=e_{s_1+1}\otimes I_{S_2}\otimes\beta I_m\ N_{39}=I_{(S_1+1)-(s_1+1)}\otimes I_{S_2}\otimes D_0-(\phi+\sigma_2)I_m$

$B_0^{78}=\begin{bmatrix} 0 & 0\\ I_{S_1}\otimes K_{24} & 0\end{bmatrix}, B_0^{87}=I_{S_1+1}\otimes K_{25},$

$K_{24}=I_{S_1+1}\otimes I_{S_2}\otimes\gamma_2\otimes\sigma_2 I_m, B_0^{810}=I_{M+1}\otimes I_{(S_1+1)S_2}\otimes\phi I_m,$

$K_{25}=I_{S_1+1}\otimes U2^0\otimes I_m,$

$$B_0^{88}=I_{S_1+1}\otimes K_{26}, K_{26}=\begin{bmatrix} N_{40} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{41}\\ \mathbf{0} & N_{40} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{41}\\ \mathbf{0} & \mathbf{0} & N_{40} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{41}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{40} & \mathbf{0} & \dots & \mathbf{0} & N_{41}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{42} & \dots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{42} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{42} \end{bmatrix},$$

where $N_{40}=I_{s_1+1}\otimes I_{S_2}\otimes(U2\oplus D_0)-(\phi+\beta)I_{n_2m}, N_{41}=e_{s_1+1}\otimes I_{S_2}\otimes\beta I_{n_2m}$

$N_{42}=I_{(S_1+1)-(s_1+1)}\otimes I_{S_2}\otimes(U2\otimes D_0)-\phi I_{n_2m},$

$B_0^{91}=I_{M+1}\otimes I_{S_1+1}\otimes\eta I_m,$

$$B_0^{99}=\begin{bmatrix} K_{27} & 0\\ 0 & I_{S_1-1}\otimes K_{28}\end{bmatrix}, K_{27}=\begin{bmatrix} N_{43} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{44}\\ \mathbf{0} & N_{43} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{44}\\ \mathbf{0} & \mathbf{0} & N_{43} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{44}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{43} & \mathbf{0} & \dots & \mathbf{0} & N_{44}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{45} & \dots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{45} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{45} \end{bmatrix},$$

where $N_{43}=I_{s_1+1}\otimes I_{S_2}\otimes D_0-(\eta+\beta)I_m, N_{44}=e_{s_1+1}\otimes I_{S_2}\otimes\beta I_m$
$N_{45}=I_{(S_1+1)-(s_1+1)}\otimes I_{S_2}\otimes D_0-\eta I_m,$

$$K_{28} = \begin{bmatrix} N_{46} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{47} \\ \mathbf{0} & N_{46} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{47} \\ \mathbf{0} & \mathbf{0} & N_{46} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{47} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{46} & \mathbf{0} & \dots & \mathbf{0} & N_{47} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{48} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{48} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{48} \end{bmatrix}, B_0^{910} = \begin{bmatrix} & 0 & \\ I_{S_1} \otimes K_{29} & & 0 \end{bmatrix}$$

where $N_{46} = I_{s_1+1} \otimes I_{S_2} \otimes D_0 - (\eta + \beta + \sigma_2)I_m, N_{47} = e_{s_1+1} \otimes I_{S_2} \otimes \beta I_m N_{48} = I_{(S_1+1)-(s_1+1)} \otimes I_{S_2} \otimes D_0 - \eta I_m, K_{29} = I_M \otimes I_{S_1+1} \otimes I_{S_2} \otimes \gamma_2 \otimes \sigma_2 I_m$, $B_0^{102} = I_{M+1} \otimes I_{(S_1+1)S_2} \otimes I_{n_2} \otimes \eta I_m$,

$$B_0^{911} = I_{S_1} \otimes K_{30}, K_{30} = \begin{bmatrix} 0 \\ I_{S_1} \otimes N_{49} \end{bmatrix}, N_{49} = I_{S_1} \otimes I_{S_2} \otimes \gamma_1 \otimes D_1,$$

$$B_0^{109} = \begin{bmatrix} K_{31} & 0 \\ & I_{S_1-1} \otimes K_{31} \end{bmatrix}, B_0^{1010} = I_{S_1} \otimes K_{32}, K_{31} = I_{M+1} \otimes I_{S_1+1} \otimes U2^0 \otimes I_m,$$

$$K_{32} = \begin{bmatrix} N_{50} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{51} \\ \mathbf{0} & N_{50} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{51} \\ \mathbf{0} & \mathbf{0} & N_{50} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{51} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{50} & \mathbf{0} & \dots & \mathbf{0} & N_{51} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{52} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{52} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{52} \end{bmatrix}, B_0^{1112} = \begin{bmatrix} & 0 & \\ I_{S_1-1} \otimes K_{37} & & 0 \end{bmatrix},$$

where $N_{49} = I_{s_1+1} \otimes I_{S_2} \otimes (U2 \oplus D_0) - (\eta + \beta)I_{n_2m}, N_{50} = e_{s_1+1} \otimes I_{S_2} \otimes \beta I_{n_2m} N_{51} = I_{(S_1+1)-(s_1+1)} \otimes I_{S_2} \otimes (U2 \oplus D_0) - \eta I_{n_2m}, K_{37} = I_{S_1S_2} \otimes I_{n_1} \otimes \gamma_2 \otimes \sigma_2 I_m$, $B_0^{113} = I_{M+1} \otimes I_{S_1S_2} \otimes I_{n_1} \otimes \eta I_m$

$$B_0^{119} = \begin{bmatrix} K_{33} & K_{34} & 0 & \dots & 0 & 0 \\ 0 & K_{33} & K_{34} & \dots & 0 & 0 \\ 0 & 0 & K_{33} & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & K_{33} & K_{34} \\ 0 & 0 & 0 & \dots & 0 & K_{33} + K_{34} \end{bmatrix}, B_0^{1111} = \begin{bmatrix} K_{35} & 0 \\ 0 & I_{S_1-1} \otimes K_{36} \end{bmatrix},$$

where $K_{33} = \begin{bmatrix} I_{S_1S_2} \otimes f_1 \theta U1^0 \otimes I_m & 0 \end{bmatrix}, K_{34} = \begin{bmatrix} I_{S_1S_2} \otimes f_2 \theta U1^0 \otimes I_m & 0 \end{bmatrix}, B_0^{124} = I_{M+1} \otimes I_{S_1S_2} \otimes I_{n_1} \otimes \gamma_1 \otimes \eta I_m$

$$K_{35}=\begin{bmatrix} N_{53} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{54} \\ \mathbf{0} & N_{53} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{54} \\ \mathbf{0} & \mathbf{0} & N_{53} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{54} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{53} & \mathbf{0} & \dots & \mathbf{0} & N_{54} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{55} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{55} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{55} \end{bmatrix}, K_{36}=\begin{bmatrix} N_{56} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{51} \\ \mathbf{0} & N_{56} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{51} \\ \mathbf{0} & \mathbf{0} & N_{56} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{51} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{56} & \mathbf{0} & \dots & \mathbf{0} & N_{51} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{57} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{57} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{57} \end{bmatrix},$$

where $N_{53}=I_{s_1}\otimes I_{S_2}\otimes(\theta U1\oplus D_0)-(\eta+\beta)I_{n_1m}, N_{54}=e_{s_1}\otimes I_{S_2}\otimes\beta I_{n_1m}$

$N_{55}=I_{(S_1-s_1)}\otimes I_{S_2}\otimes(\theta U1\oplus D_0)-\eta I_{n_1m}, N_{56}=I_{s_1}\otimes I_{S_2}\otimes(\theta U1\oplus D_0)-(\eta+\sigma_2+\beta)I_{n_1m},$

$N_{57}=I_{(S_1-s_1)}\otimes I_{S_2}\otimes(\theta U1\oplus D_0)-(\sigma+\eta)I_{n_1m},$

$$B_0^{1210}=\begin{bmatrix} K_{38} & K_{39} & 0 & \dots & 0 & 0 \\ 0 & K_{38} & K_{39} & \dots & 0 & 0 \\ 0 & 0 & K_{38} & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & K_{38} & K_{39} \\ 0 & 0 & 0 & \dots & 0 & K_{38}+K_{39} \end{bmatrix},$$

where $K_{38}=\left[I_{S_1S_2}\otimes f_1\theta U1^0\otimes I_{n_2}\otimes I_m\ 0\right], K_{39}=\left[I_{S_1S_2}\otimes f_2\theta U1^0\otimes I_{n_2}\otimes I_m\ 0\right],$

$B_0^{1211}=I_{S_1}\otimes K_{40}, K_{40}=\begin{bmatrix} N_{58} & 0 \\ I_{S_1-1}\otimes N_{58} & 0\end{bmatrix},$

$$B_0^{1212}=I_{S1}\otimes K_{41}, K_{41}=\begin{bmatrix} N_{59} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{60} \\ \mathbf{0} & N_{59} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{60} \\ \mathbf{0} & \mathbf{0} & N_{59} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{60} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{59} & \mathbf{0} & \dots & \mathbf{0} & N_{60} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{61} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{61} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{61} \end{bmatrix},$$

where $N_{58}=I_{S_1}\otimes I_{n_1}\otimes U2^0\otimes I_m, N_{59}=I_{s_1}\otimes I_{S_2}\otimes((\theta U1+U2)\oplus D_0)-\eta I_{n_1n_2m},$

$N_{60}=e_{s_1}\otimes I_{S_2}\otimes\beta I_{n_1n_2m}\ N_{61}=I_{(S_1-s_1)}\otimes I_{S_2}\otimes((\theta U1+U2)\oplus D_0)-\eta I_{n_1n_2m},$

$$A_0=\begin{bmatrix} A_0^{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_0^{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_0^{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_0^{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_0^{55} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_0^{66} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_0^{77} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_0^{88} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_0^{99} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_0^{1010} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_0^{1111} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_0^{1212} \end{bmatrix},$$

$A_0^{11}=I_{S_1+1}\otimes K_{42}, A_0^{22}=I_{S_1+1}\otimes K_{43},$

$K_{42}=\begin{bmatrix} I_{S_2}\otimes D_1 & 0\\ 0 & 0\end{bmatrix}, K_{43}=\begin{bmatrix} I_{S_2n_2}\otimes D_1 & 0\\ 0 & 0\end{bmatrix},$
$A_0^{33}=I_{M+1}\otimes I_{S_1S_2}\otimes I_{n_1\otimes D_1}, A_0^{44}=I_{M+1}\otimes I_{S_1S_2}\otimes I_{n_1n_2}\otimes D_1,$
$A_0^{55}=I_{M+1}\otimes I_{S_1S_2}\otimes D_1, A_0^{66}=I_{M+1}\otimes I_{S_1S_2}\otimes I_{n_1n_2}\otimes D_1,$
$A_0^{77}=I_{M+1}\otimes I_{(S_1+1)S_2}\otimes D_1, A_0^{88}=I_{M+1}\otimes I_{(S_1+1)S_2}\otimes I_{n_2}\otimes D_1,$
$A_0^{99}=I_{M+1}\otimes I_{(S_1+1)S_2}\otimes D_1, A_0^{1010}=I_{M+1}\otimes I_{(S_1+1)S_2}\otimes I_{n_2}\otimes D_1,$
$A_0^{1111}=I_{M+1}\otimes I_{S_1S_2}\otimes I_{n_1}\otimes D_1, A_0^{1212}=I_{M+1}\otimes I_{S_1S_2}\otimes I_{n_1n_2}\otimes D_1,$

$$A_2=\begin{bmatrix}
0&0&A_2^{13}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&A_2^{24}&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&A_2^{911}&0\\
0&0&0&0&0&0&0&0&0&0&0&A_2^{1012}\\
0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}, A_2^{13}=I_{S_1+1}\otimes K_{44},$$

$K_{44}=\begin{bmatrix} 0\\ I_{S_1}\otimes I_{S_2}\otimes\gamma_1\otimes\sigma_1 I_m\end{bmatrix}, A_2^{24}=I_{S_1+1}\otimes K_{45}\ A_2^{911}=I_{S_1}\otimes K_{46},$

$K_{45}=\begin{bmatrix} 0\\ I_{S_1}\otimes I_{S_2}\otimes\gamma_2\otimes\sigma_1 I_m\end{bmatrix}, K_{46}=\begin{bmatrix} 0\\ I_{S_1}\otimes I_{S_2}\otimes\gamma_1\otimes\sigma_1 I_m\end{bmatrix},$

$A_2^{1012}=I_{S_1}\otimes K_{47}, A_1^{11}=\begin{bmatrix} K_{48} & 0\\ 0 & I_{S_1-1}\otimes K_{49}\end{bmatrix}, K_{47}=\begin{bmatrix} 0\\ I_{S_1S_2}\otimes I_{n_1}\otimes\gamma_2\otimes\sigma_1 I_m\end{bmatrix},$

$$K_{48}=\begin{bmatrix}
N_{62}&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&N_{63}\\
\mathbf{0}&N_{64}&\mathbf{0}&\cdots&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&N_{63}\\
\mathbf{0}&\mathbf{0}&N_{64}&\cdots&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&N_{63}\\
\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\vdots\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots&N_{64}&\mathbf{0}&\cdots&\mathbf{0}&N_{63}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&N_{65}&\cdots&\mathbf{0}&\mathbf{0}\\
\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\vdots\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&\mathbf{0}&\cdots&N_{65}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&N_{65}
\end{bmatrix}, K_{49}=\begin{bmatrix}
N_{66}&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&N_{67}\\
\mathbf{0}&N_{68}&\mathbf{0}&\cdots&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&N_{67}\\
\mathbf{0}&\mathbf{0}&N_{68}&\cdots&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&N_{67}\\
\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots&\vdots&\vdots\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots&N_{68}&\mathbf{0}&\cdots&\mathbf{0}&N_{67}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&N_{69}&\cdots&\mathbf{0}&\mathbf{0}\\
\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\ddots&\vdots&\vdots\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&\mathbf{0}&\cdots&N_{69}&\mathbf{0}\\
\mathbf{0}&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&\mathbf{0}&\cdots&\mathbf{0}&N_{69}
\end{bmatrix},$$

where $N_{62}=I_{S_2}\otimes D_0-\beta I_m, N_{63}=e_{s_1}\otimes I_{S_2}\otimes\beta I_m\ N_{64}=I_{S_2}\otimes D_0-(\beta+\sigma_1)I_m,$
$N_{65}=I_{S_2}\otimes D_0-\sigma_1 I_m, N_{66}=I_{S_2}\otimes D_0-(\beta+\sigma_2)I_m, N_{67}=e_{s_1}\otimes I_{S_2}\otimes\beta I_m$
$N_{68}=I_{S_2}\otimes D_0-(\beta+\sigma_1+\sigma_2)I_m, N_{69}=I_{S_2}\otimes D_0-(\sigma_1+\sigma_2)I_m,$

$$A_1 = \begin{bmatrix} A_1^{11} & A_1^{12} & A_1^{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_1^{21} & A_1^{22} & 0 & A_1^{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_1^{31} & 0 & A_1^{33} & A_1^{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_1^{42} & A_1^{43} & A_1^{44} & 0 & A_1^{46} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_1^{53} & 0 & A_1^{55} & A_1^{56} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A_1^{64} & A_1^{65} & A_1^{66} & 0 & 0 & 0 & 0 & 0 & 0 \\ A_1^{71} & 0 & 0 & 0 & 0 & 0 & A_1^{77} & A_1^{78} & 0 & 0 & 0 & 0 \\ 0 & A_1^{82} & 0 & 0 & 0 & 0 & A_1^{87} & A_1^{88} & 0 & 0 & 0 & 0 \\ A_1^{91} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_1^{99} & A_1^{910} & A_1^{911} & 0 \\ 0 & A_1^{102} & 0 & 0 & 0 & 0 & 0 & 0 & A_1^{109} & A_1^{1010} & 0 & A_1^{1012} \\ 0 & 0 & A_1^{113} & 0 & 0 & 0 & 0 & 0 & A_1^{119} & 0 & A_1^{1111} & A_1^{1112} \\ 0 & 0 & 0 & A_1^{124} & 0 & 0 & 0 & 0 & 0 & A_1^{1210} & A_1^{1211} & A_1^{1212} \end{bmatrix},$$

$A_1^{12} = \begin{bmatrix} 0 \\ I_{S_1+1} \otimes K_{50} \end{bmatrix}, A_1^{13} = I_{S_1+1} \otimes K_{51},$

$K_{50} = I_{S_1+1} \otimes I_{S_2} \otimes \gamma_2 \otimes \sigma_2 I_m, K_{51} = \begin{bmatrix} 0 \\ I_{S_2 S_2} \otimes D_1 \end{bmatrix},$

$A_1^{21} = I_{S_1+1} \otimes K_{52}, K_{52} = \begin{bmatrix} N_{70} & 0 \\ I_{S_1} \otimes N_{70} & 0 \end{bmatrix}, N_{70} = I_{S_1+1} \otimes U2^0 \otimes I_m,$

$$A_1^{22} = I_{S_1+1} \otimes K_{53}, K_{53} = \begin{bmatrix} N_{71} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{72} \\ \mathbf{0} & N_{73} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{72} \\ \mathbf{0} & \mathbf{0} & N_{73} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{72} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{73} & \mathbf{0} & \dots & \mathbf{0} & N_{72} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{74} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{74} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{74} \end{bmatrix},$$

where $N_{71} = I_{S_2} \otimes (U2 \oplus D_0) - \beta I_{n_2}, N_{72} = e_{s_1} \otimes I_{S_2} \otimes I_{n_2} \otimes \beta I_m$

$N_{73} = I_{S_2} \otimes (U2 \oplus D_0) - (\beta + \sigma_1) I_{n_2 m}, N_{74} = I_{S_2} \otimes (U_2 + D_0) - \sigma_1 I_{n_2 m},$

$$A_1^{24} = I_{S_1+1} \otimes K_{54}, A_1^{31} = \begin{bmatrix} K_{55} & K_{56} & 0 & \dots & 0 & 0 \\ 0 & K_{55} & K_{56} & \dots & 0 & 0 \\ 0 & 0 & K_{55} & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & K_{55} & K_{56} \\ 0 & 0 & 0 & \dots & 0 & K_{55} + K_{56} \end{bmatrix},$$

$K_{54} = \begin{bmatrix} 0 \\ I_{S_1 S_2} \otimes I_{n_2} \otimes \gamma_1 \otimes D_1 \end{bmatrix}, K_{55} = \begin{bmatrix} I_{S_1 S_2} \otimes f_1 U1^0 \otimes I_m & 0 \end{bmatrix},$

$K_{56} = \begin{bmatrix} I_{S_1 S_2} \otimes f_2 U1^0 \otimes I_m & 0 \end{bmatrix},$

$$A_1^{33} = \begin{bmatrix} K_{57} & 0 \\ 0 & I_{S_1-1} \otimes K_{58} \end{bmatrix}, K_{57} = \begin{bmatrix} N_{75} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{76} \\ \mathbf{0} & N_{75} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{76} \\ \mathbf{0} & \mathbf{0} & N_{75} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{76} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{75} & \mathbf{0} & \dots & \mathbf{0} & N_{76} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{77} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{77} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{77} \end{bmatrix},$$

where $N_{75} = I_{s_1} \otimes I_{S_2} \otimes (U1 \oplus D_0) - \beta I_{n_1 m}, N_{76} = e_{s_1} \otimes I_{S_2} \otimes I_{n_1} \otimes \beta I_m$

$N_{77} = I_{(S_1-s_1)} \otimes I_{S_2} \otimes (U1 \oplus D_0),$

$$K_{58} = \begin{bmatrix} N_{78} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{79} \\ \mathbf{0} & N_{78} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{79} \\ \mathbf{0} & \mathbf{0} & N_{78} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{79} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{78} & \mathbf{0} & \dots & \mathbf{0} & N_{79} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{80} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{80} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{80} \end{bmatrix}, A_1^{34} = \begin{bmatrix} & 0 \\ I_{S_1-1} \otimes K_{59} & 0 \end{bmatrix},$$

$K_{59} = [I_{S_1S_2} \otimes I_{n1} \otimes \gamma_2 \otimes \sigma_2 I_m],$

$$A_1^{42} = \begin{bmatrix} K_{60} & K_{61} & 0 & \dots & 0 & 0 \\ 0 & K_{60} & K_{61} & \dots & 0 & 0 \\ 0 & 0 & K_{60} & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & K_{60} & K_{61} \\ 0 & 0 & 0 & \dots & 0 & K_{60}+K_{61} \end{bmatrix}, A_1^{43} = I_{S_1} \otimes K_{62},$$

where $N_{78} = I_{s_1} \otimes I_{S_2} \otimes (U1 \oplus D_0) - (\beta + \sigma_2) I_{n_1 m}, N_{79} = e_{s_1} \otimes I_{S_2} \otimes I_{n_1} \otimes \beta I_m N_{80} = I_{(S_1-s_1)} \otimes I_{S_2} \otimes (U1 \oplus D_0) - \sigma_2 I_{n_1 m},$
$K_{60} = \left[I_{S_1} \otimes I_{S_2} \otimes f_1 U1^0 \otimes I_{n_2 m} \quad 0\right], K_{61} = \left[I_{S_1} \otimes I_{S_2} \otimes f_2 U1^0 \otimes I_{n_2 m} \quad 0\right], K_{62} = \begin{bmatrix} N_{81} & 0 \\ I_{S_1-1} \otimes N_{81} & \end{bmatrix}, A_1^{44} = I_{S_1-1} \otimes K_{63}, N_{81} = I_{S_1} \otimes I_{n_1} \otimes U2^0 \otimes I_m,$

$$K_{63} = \begin{bmatrix} N_{82} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{83} \\ \mathbf{0} & N_{82} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{83} \\ \mathbf{0} & \mathbf{0} & N_{82} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{83} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{82} & \mathbf{0} & \dots & \mathbf{0} & N_{83} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{84} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{84} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{84} \end{bmatrix} A_1^{56} = \begin{bmatrix} & 0 \\ I_{S_1} \otimes K_{65} & 0 \end{bmatrix},$$

$A_1^{53} = I_{M+1} \otimes I_{S_1} \otimes I_{S_2} \otimes \gamma_1 \otimes \delta I_m, K_{65} = I_{S_1S_2} \otimes \gamma_2 \otimes \sigma_2 I_m,\ A_1^{64} = I_{M+1} \otimes I_{S_1S_2} \otimes \gamma_1 \otimes I_{n_2} \otimes \delta I_m,$

$$A_1^{55} = I_{S_1} \otimes K_{64}, K_{64} = \begin{bmatrix} N_{85} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{86} \\ \mathbf{0} & N_{85} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{86} \\ \mathbf{0} & \mathbf{0} & N_{85} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{86} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{85} & \mathbf{0} & \dots & \mathbf{0} & N_{86} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{87} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{87} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{87} \end{bmatrix}$$

$N_{85} = I_{s_1} \otimes I_{S_2} \otimes D_0 - (\delta + \beta) I_m, N_{86} = e_{s_1} \otimes I_{S_2} \otimes \beta I_m$

$$N_{87} = I_{(S_1-s_1)} \otimes I_{S_2} \otimes D_0 - \delta I_m,$$

$$A_1^{65} = I_{S_1} \otimes K_{66}, K_{66} = \begin{bmatrix} N_{88} & 0 \\ I_{S_1-1} \otimes N_{88} & \end{bmatrix},$$

where $N_{88} = I_{S_1} \otimes U2^0 \otimes I_m, N_{85} = I_{s_1} \otimes I_{S_2} \otimes ((U1 + U2) \oplus D_0) - (\tau + \beta) I_{n_1 n_2 m}, N_{86} = e_{s_1} \otimes I_{S_2} \otimes I_{n_1 n_2} \otimes \beta I_m\ N_{87} = I_{(S_1-s_1)} \otimes I_{S_2} \otimes ((U1 + U2) \oplus D_0) - \tau I_{n_1 n_2 m}, A_1^{46} = I_{M+1} \otimes I_{S_1} \otimes I_{S_2} \otimes e_{n_1} \otimes \tau I_{n_2 m},$

$$A_1^{66} = I_{S_1} \otimes K_{67}, K_{67} = \begin{bmatrix} N_{89} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{90} \\ \mathbf{0} & N_{89} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{90} \\ \mathbf{0} & \mathbf{0} & N_{89} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{90} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{89} & \mathbf{0} & \dots & \mathbf{0} & N_{90} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{91} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{91} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{91} \end{bmatrix},$$

where $N_{89} = I_{s_1} \otimes I_{S_2} \otimes (U2 \oplus D_0) - (\delta + \beta)I_{n_2m}, N_{90} = e_{s_1} \otimes I_{S_2} \otimes I_{n_2} \otimes \beta I_m$
$N_{88} = I_{(S_1-s_1)} \otimes I_{S_2} \otimes (U2 \oplus D_0) - \delta In_2m,$
$A_1^{71} = I_{M+1} \otimes I_{(S_1+1)S_2} \otimes \phi I_m,$

$$A_1^{77} = I_{S_1+1} \otimes K_{68}, K_{68} = \begin{bmatrix} N_{92} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{93} \\ \mathbf{0} & N_{92} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{93} \\ \mathbf{0} & \mathbf{0} & N_{92} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{93} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{92} & \mathbf{0} & \dots & \mathbf{0} & N_{93} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{94} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{94} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{94} \end{bmatrix},$$

where $N_{92} = I_{s_1+1} \otimes I_{S_2} \otimes D_0 - (\phi + \sigma_2)I_m, N_{93} = e_{s_1+1} \otimes I_{S_2} \otimes \beta I_m$
$N_{94} = I_{(S_1+1)-(s_1+1)} \otimes I_{S_2} \otimes D_0 - (\phi + \sigma_2)Im,$

$$K_{69} = \begin{bmatrix} N_{95} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{96} \\ \mathbf{0} & N_{95} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{96} \\ \mathbf{0} & \mathbf{0} & N_{95} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{96} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{95} & \mathbf{0} & \dots & \mathbf{0} & N_{96} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{97} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{97} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{97} \end{bmatrix}, A_1^{78} = \begin{bmatrix} 0 \\ I_{S_1} \otimes K_{70} \end{bmatrix},$$

where $N_{95} = I_{s_1+1} \otimes I_{S_2} \otimes D_0 - (\phi + \beta + \sigma_2)I_m, N_{96} = e_{s_1+1} \otimes I_{S_2} \otimes \beta I_m$
$N_{97} = I_{(S_1+1)-(s_1+1)} \otimes I_{S_2} \otimes D_0 - (\phi + \sigma_2)Im, K_{70} = I_{S_1+1} \otimes I_{S_2} \otimes \gamma_2 \otimes \sigma_2 I_m$, $A_1^{82} = I_{M+1} \otimes I_{(S_1+1)S_2n_2 \otimes \phi Im}$,
$A_1^{87} = I_{S_1+1} \otimes K_{71}, K_{71} = \begin{bmatrix} N_{98} & 0 \\ I_{S_1} \otimes N_{98} & 0 \end{bmatrix}, N_{98} = I_{S_1+1} \otimes U2^0 \otimes I_m,$

$$A_1^{88} = I_{S_1} \otimes K_{72}, K_{72} = \begin{bmatrix} N_{99} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{100} \\ \mathbf{0} & N_{99} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{100} \\ \mathbf{0} & \mathbf{0} & N_{99} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{100} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{99} & \mathbf{0} & \dots & \mathbf{0} & N_{100} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{101} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{101} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{101} \end{bmatrix},$$

where $N_{99} = I_{s_1+1} \otimes I_{S_2} \otimes (U2 \oplus D_0) - (\phi + \beta)I_{n_2m}, N_{100} = e_{s_1+1} \otimes I_{S_2} \otimes \beta I_{n_2m} N_{101} = I_{(S_1+1)-(s_1+1)} \otimes I_{S_2} \otimes (U2 \oplus D_0) - \phi In_2m$, $A_1^{91} = I_{M+1} \otimes I_{S_1+1}S_2 \otimes \eta I_m,$

$$A_1^{99} = \begin{bmatrix} K_{73} & 0 \\ 0 & I_{S_1-1} \otimes K_{74} \end{bmatrix}, K_{73} = \begin{bmatrix} N_{102} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{103} \\ \mathbf{0} & N_{104} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{103} \\ \mathbf{0} & \mathbf{0} & N_{104} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{103} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{104} & \mathbf{0} & \dots & \mathbf{0} & N_{103} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{105} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{105} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{105} \end{bmatrix},$$

where $N_{102} = I_{s_1+1} \otimes I_{S_2} \otimes (U2 \oplus D_0) - (\phi + \beta)I_{n_2m}, N_{103} = e_{s_1} \otimes I_{S_2} \otimes \beta I_m$

$N_{104} = I_{s_1} \otimes I_{S_2} \otimes D_0 - (\eta + \beta + \sigma_1)I_m, N_{105} = I_{(S_1+1)-(s_1+1)} \otimes I_{S_2} \otimes D_0 - (\eta + \sigma_1)I_m,$

$$K_{74} = \begin{bmatrix} N_{106} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{107} \\ \mathbf{0} & N_{108} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{107} \\ \mathbf{0} & \mathbf{0} & N_{108} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{107} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{108} & \mathbf{0} & \dots & \mathbf{0} & N_{107} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{109} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{109} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{109} \end{bmatrix}, A_1^{910} = \begin{bmatrix} 0 & \\ I_{S_1-1} \otimes K_{75} & 0 \end{bmatrix},$$

where $N_{106} = I_{S_2} \otimes D_0 - (\eta + \beta + \sigma_2)I_m, N_{107} = e_{S_1} \otimes I_{S_2} \otimes \beta I_m$

$N_{108} = I_{s_1} \otimes I_{S_2} \otimes D_0 - (\eta + \beta + \sigma_1)I_m, N_{109} = I_{(S_1+1)-(s_1+1)} \otimes I_{S_2} \otimes D_0 - (\eta + \sigma_1)I_m,$

$K_{75} = I_{S_1} \otimes I_{(S_1+1)S_2} \otimes \gamma_2 \otimes \sigma_2 I_m,$

$A_1^{911} = I_{S_1} \otimes K_{76}, A_1^{109} = I_{S_1} \otimes K_{77}, A_1^{102} = I_{M+1} \otimes I_{(S_1+1)S_2} \otimes I_{n_2} \otimes \eta I_m,$

$$K_{76} = \begin{bmatrix} 0 \\ I_{S_1} \otimes I_{S_2} \otimes \gamma_1 \otimes D_1 \end{bmatrix}, K_{77} = \begin{bmatrix} N_{110} & 0 \\ I_{S_1-1} \otimes N_{110} & 0 \end{bmatrix},$$

$$A_1^{1010} = I_{S_1-1} \otimes K_{78}, K_{78} = \begin{bmatrix} N_{111} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{112} \\ \mathbf{0} & N_{113} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{112} \\ \mathbf{0} & \mathbf{0} & N_{113} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{112} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{113} & \mathbf{0} & \dots & \mathbf{0} & N_{112} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{114} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{114} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{114} \end{bmatrix},$$

where $N_{110} = I_{(S_1+1)} \otimes U2^0 \otimes I_m,$

$N_{111} = I_{S_2} \otimes (U2 \oplus D_0) - (\eta + \beta)I_{n_2m}, N_{112} = e_{s_1} \otimes I_{S_2} \otimes \beta I_{n_2m}$
$N_{113} = I_{S_1} \otimes I_{S_2} \otimes (U2 \oplus D_0) - (\eta + \beta + \sigma_1)I_{n_2m}, N_{114} = I_{(S_1+1)-(s_1+1)} \otimes I_{S_2} \otimes (U2 \oplus D_0) - (\eta + \sigma_1)I_{n_2m},$

$$A_1^{1012} = I_{S_1} \otimes K_{79}, K_{79} = \begin{bmatrix} 0 \\ I_{S_1} \otimes I_{S_2} \otimes \gamma_1 \otimes I_{n_2} \otimes D_1 \end{bmatrix},$$

$$A_1^{1109} = \begin{bmatrix} K_{80} & K_{81} & 0 & \dots & 0 & 0 \\ 0 & K_{80} & K_{81} & \dots & 0 & 0 \\ 0 & 0 & K_{80} & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & K_{80} & K_{81} \\ 0 & 0 & 0 & \dots & 0 & K_{80} + K_{81} \end{bmatrix}, A_1^{1111} = \begin{bmatrix} K_{82} & 0 \\ 0 & I_{S_1-1} \otimes K_{83} \end{bmatrix},$$

$K_{80} = [I_{S_1S_2} \otimes f_1 \theta U1^0 \otimes I_m \; 0], K_{81} = [I_{S_1S_2} \otimes f_2 \theta U1^0 \otimes I_m \; 0],$

$$K_{82}=\begin{bmatrix} N_{115} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{116} \\ \mathbf{0} & N_{115} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{116} \\ \mathbf{0} & \mathbf{0} & N_{115} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{116} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{115} & \mathbf{0} & \dots & \mathbf{0} & N_{116} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{117} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{117} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{117} \end{bmatrix}, K_{83}$$
$$=\begin{bmatrix} N_{118} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{119} \\ \mathbf{0} & N_{118} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{119} \\ \mathbf{0} & \mathbf{0} & N_{118} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{119} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{118} & \mathbf{0} & \dots & \mathbf{0} & N_{119} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{120} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{120} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{120} \end{bmatrix},$$

where $N_{115} = I_{s_1} \otimes I_{S_2} \otimes (\theta U1 \oplus D_0) - (\eta + \beta) I_{n_1 m}, N_{116} = e_{s_1} \otimes I_{S_2} \otimes \beta I_{n_1 m}$
$N_{117} = I_{(S_1 - s_1)} \otimes I_{S_2} \otimes (\theta U1 \oplus D_0) - \eta I_{n_1 m}, N_{118} = I_{s_1} \otimes I_{S_2} \otimes (\theta U1 \oplus D_0) - (\eta + \beta + \sigma_2) I_{n_1 m}$,
$N_{119} = e_{s_1} \otimes I_{S_2} \otimes \beta I_{n_1 m}$, $N_{120} = I_{S_1 - s_1} \otimes I_{S_2} \otimes (\theta U1 \oplus D_0) - (\eta + \beta + \sigma_2) I_{n_1 m}$,
$A_{124} = I_{M+1} \otimes I_{S_1 S_2} \otimes I_{n_2} \otimes e_{n_1} \otimes \gamma_1 \otimes \eta I_m$
, $A_1^{1112} = \begin{bmatrix} 0 \\ I_{S_1 - 1} \otimes K_{84} & 0 \end{bmatrix}, K_{84} = I_{S_1 S_2} \otimes I_{n_1} \otimes \gamma_2 \otimes \sigma_2 I_{n_1 m}$,

$$A_1^{1210} = \begin{bmatrix} K_{85} & K_{86} & 0 & \dots & 0 & 0 \\ 0 & K_{85} & K_{86} & \dots & 0 & 0 \\ 0 & 0 & K_{85} & \dots & 0 & 0 \\ & & & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & K_{85} & K_{86} \\ 0 & 0 & 0 & \dots & 0 & K_{85} + K_{86} \end{bmatrix}, A_1^{1211} = I_{S_1} \otimes K_{87},$$

$K_{85} = \begin{bmatrix} I_{S_1 S_2} \otimes f_1 \theta U1^0 \otimes I_{n_2 m} & 0 \end{bmatrix}, K_{86} = \begin{bmatrix} I_{S_1 S_2} \otimes f_2 \theta U1^0 \otimes I_{n_2 m} & 0 \end{bmatrix}$,
$K_{87} = \begin{bmatrix} N_{121} & 0 \\ I_{S_1 - 1} \otimes N_{121} & 0 \end{bmatrix}, N_{121} = I_{S_1} \otimes U2^0 \otimes I_{n_1 m}$

,

$$A_1^{1212} = I_{S_1} \otimes K_{88}, K_{88} = \begin{bmatrix} N_{122} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{123} \\ \mathbf{0} & N_{122} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{123} \\ \mathbf{0} & \mathbf{0} & N_{122} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{123} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & N_{122} & \mathbf{0} & \dots & \mathbf{0} & N_{123} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{124} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & N_{124} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & N_{124} \end{bmatrix},$$

$N_{122} = I_{s_1} \otimes I_{S_2} \otimes (\theta U1 \otimes D_0) - (\eta + \beta) I_{n_1 n_2 m}$,
$N_{124} = I_{(S_1 - s_1)} \otimes I_{S_2} \otimes ((U1 + U2) \oplus N_0) - \eta I_{n_1 n_2 m}$.

4.ANALYSIS OF STABILITY CONDITION

A generator matrix is indicated by the equation $A = A_0 + A_1 + A_2$. In the steady state, let ζ be the probability vector of A that satisfies $\zeta e = 1$ and $\zeta A = 0$. Ühere $\zeta_0, \zeta_6, \zeta_8$, has a dimension of $(M+1)(S_1+1)S_2 m$, $\zeta_1, \zeta_7, \zeta_9$, has a dimension of $(M+1)(S_1+1)S_2 n_2 m$, ζ_2, has a dimension of $(M+1)S_1S_2n_1m$, $\zeta_3, \zeta_{10}, \zeta_{11}$, has a dimension of $(M+1)S_1S_2n_1n_2m$, ζ_4 has a dimension of $(M+1)S_1S_2m, \zeta_5$ has a dimension of $(M+1)S_1S_2n_2m$.

$$A = \begin{bmatrix} A^{11} & A^{12} & A^{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A^{21} & A^{22} & 0 & 0 & A^{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A^{31} & 0 & A^{33} & A^{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A^{42} & A^{43} & A^{44} & 0 & A^{46} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A^{55} & A^{56} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A^{64} & A^{65} & A^{66} & 0 & 0 & 0 & 0 & 0 \\ A^{71} & 0 & 0 & 0 & 0 & 0 & A^{77} & A^{78} & 0 & 0 & 0 & 0 \\ 0 & A^{82} & 0 & 0 & 0 & 0 & A^{87} & A^{88} & 0 & 0 & 0 & 0 \\ A^{91} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A^{99} & A^{910} & A^{911} & 0 \\ 0 & A^{102} & 0 & 0 & 0 & 0 & 0 & 0 & A^{109} & A^{1010} & 0 & A^{1012} \\ 0 & 0 & A^{113} & 0 & 0 & 0 & 0 & 0 & A^{119} & 0 & A^{1111} & A^{1112} \\ 0 & 0 & 0 & A^{124} & 0 & 0 & 0 & 0 & A^{1210} & A^{1211} & 0 & A^{1212} \end{bmatrix},$$

where, $A^{11} = [A_0^{11} + A_1^{11}], A^{12} = A_1^{12}, A^{13} = [A_1^{13} + A_2^{13}], A^{21} = A_1^{21}$,
$A^{22} = [A_0^{22} + A_1^{22}], A^{24} = [A_1^{24} + A_2^{24}], A^{31} = A_1^{31}, A^{33} = [A_0^{33} + A_1^{33}]$,
$A^{34} = A_1^{34}, A^{42} = A_1^{42}, A^{43} = A_1^{43}, A^{44} = A_0^{44}, A^{46} = A_1^{46}, A^{55} = \left[A_0^{55} + A_1^{56}\right]$,
$A^{64} = A_1^{64}, A^{65} = A_1^{65}, A^{66} = [A_0^{66} + A_1^{66}], A^{71} = A_1^{71}, A^{77} = [A_0^{77} + A_1^{77}], A^{78} = A_1^{78}$,
$A^{82} = A_1^{82}, A^{87} = A_1^{87}, A^{88} = [A_0^{88} + A_1^{88}], A^{91} = A_1^{91}, A^{99} = [A_0^{99} + A_1^{99}], A^{91} = A_1^{91}$,
$A^{910} = A_1^{910}, A^{911} = [A_1^{911} + A_2^{911}], A^{102} = A_1^{102}, A^{109} = A_1^{109}, A^{1010} = [A_0^{1010} + A_1^{1010}]$,
$A^{1012} = [A_0^{1012} + A_1^{1012}], A^{113} = A_1^{113}, A^{119} = A_1^{119}, A^{1111} = [A_0^{1111} + A_1^{1111}]$,
$A^{1112} = A_1^{1112}, A^{124} = A_1^{124}, A^{1210} = A_1^{1210}, A^{1211} = A_1^{1211}, A^{1212} = [A_0^{1212} + A_1^{1212}]$,
The steady state probability vector ζ is obtained by solving the below equations:
$\zeta_0[A^{11}] + \zeta_1[A^{21}] + \zeta_2[A^{31}] + \zeta_6[A^{71}] + \zeta_8[A^{91}] = 0$,
$\zeta_0[A^{12}] + \zeta_1[A^{22}] + \zeta_3[A^{42}] + \zeta_7[A^{82}] + \zeta_9[A^{102}] = 0$,
$\zeta_0[A^{13}] + \zeta_2[A^{33}] + \zeta_3[A^{43}] + \zeta_{10}[A^{113}] = 0$,
$\zeta_1[A^{24}] + \zeta_2[A^{34}] + \zeta_3[A^{44}] + \zeta_5[A^{64}] + \zeta_{11}[A^{124}] = 0$,
$\zeta_4[A^{55}] + \zeta_5[A^{65}] = 0$,
$\zeta_3[A^{46}] + \zeta_4[A^{56}] + \zeta_5[A^{66}] = 0$,
$\zeta_7[A^{77}] + \zeta_8[A^{87}] = 0$,
$\zeta_7[A^{78}] + \zeta_8[A^{88}] = 0$,
$\zeta_8[A^{99}] + \zeta_9[A^{109}] + \zeta_{10}[A^{119}] = 0$,
$\zeta_8[A^{910}] + \zeta_9[A^{1010}] + \zeta_{11}[A^{1210}] = 0$,
$\zeta_8[A^{911}] + \zeta_{10}[A^{1111}] + \zeta_{11}[A^{1211}] = 0$,
$\zeta_9[A^{1012}] + \zeta_{10}[A^{1112}] + \zeta_{11}[A^{1212}] = 0$,
Subject to normalizing condition

$$\sum_{d_2=0}^{M} \sum_{d_5=0}^{S_1} \sum_{d_6=1}^{S_2} \left[\zeta_0 e_m + \zeta_1 e_{n_2m} + \zeta_2 e_{n_1m} + \zeta_3 e_{n_1n_2m} + \zeta_4 e_m + \zeta_5 e_{n_2m} \right.$$
$$\left. + \zeta_6 e_{n_2m} + \zeta_7 e_{n_2m} + \zeta_8 e_m + \zeta_9 e_{n_2m} + \zeta_{10} e_{n_1n_2m} + \zeta_{11} e_{n_1n_2m}\right] = 1$$

Within the framework of QBD, our model's stability should meet the necessary and sufficient conditions, (i.e.),

$$\zeta A_0 e < \zeta A_2 e \#(3)$$

which is found to be of the below form,

$$\sum_{d_2=0}^{M}\sum_{d_5=0}^{S_1}\sum_{d_6=1}^{S_2}\Big[\zeta_0 d_2 d_3 d_6\left[I_{S_2}\otimes D_1 e_m\right]+\zeta_1 d_2 d_3 d_6\left[I_{S_2 n_2}\otimes D_1 e_{n_2 m}\right]$$
$$+\zeta_2 d_2 d_3 d_6\left[I_{M+1}\otimes I_{S_1 S_2}\otimes I_{n_1}\otimes D_1 e_{n_1 m}\right]+\zeta_3 d_2 d_3 d_6\left[I_{M+1}\otimes I_{S_1 S_2}\otimes I_{n_1 n_2}\otimes D_1 e_{n_1 n_2 m}\right]$$
$$+\zeta_4 d_2 d_3 d_6\left[I_{M+1}\otimes I_{S_1 S_2}\otimes D_1 e_m\right]+\zeta_5 d_2 d_3 d_6\left[I_{M+1}\otimes I_{S_1 S_2}\otimes I_{n_1 n_2}\otimes D_1 e_{n_2 m}\right]$$
$$+\zeta_6 d_2 d_3 d_6\left[I_{M+1}\otimes I_{(S_1+1)S_2}\otimes D_1 e_m\right]+\zeta_7 d_2 d_3 d_6\left[I_{M+1}\otimes I_{(S_1+1)S_2}\otimes I_{n_2}\otimes D_1 e_{n_2 m}\right]$$
$$+\zeta_8 d_2 d_3 d_6\left[I_{M+1}\otimes I_{(S_1+1)S_2}\otimes D_1 e_m\right]+\zeta_9\left[I_{M+1}\otimes I_{(S_1+1)S_2}\otimes I_{n_2}\otimes D_1 e_{n_2 m}\right]$$
$$+\zeta_{10} d_2 d_3 d_6\left[I_{M+1}\otimes I_{S_1 S_2}\otimes I_{n_2}\otimes D_1 e_{n_1 n_2 m}\right]+\zeta_{11} d_2 d_3 d_6\left[I_{M+1}\otimes I_{S_1 S_2}\otimes I_{n_2 n_2}\otimes D_1 e_{n_1 n_2 m}\right]\Big]$$
$$<$$
$$\sum_{d_2=0}^{M}\sum_{d_5=0}^{S_1}\sum_{d_6=1}^{S_2}\Big[\zeta_0 d_2 d_3 d_6\left[I_{S_1}\otimes I_{S_2}\otimes \gamma_1\otimes \sigma_1 I_m e_m\right]+\zeta_1 d_2 d_3 d_6\left[I_{S_1}\otimes I_{S_2}\otimes \gamma_2\otimes \sigma_1\otimes I_m e_{n_2 m}\right]$$
$$+\zeta_9 d_2 d_3 d_6\left[I_{S_1 S_2}\otimes I_{n_1}\otimes \gamma_2\otimes \sigma_1 I_m e_{n_2 m}\right]+\zeta_{10} d_2 d_3 d_6\left[I_{S_1}\otimes I_{S_2}\otimes \gamma_1\otimes \sigma_1 I_m e_{n_1 n_2 m}\right]$$

4.1. Invariant Probability Vector Analysis

Let ν symbolize the infinitesimal generator Q's invariant probability vector and this is divided into $\nu=(\nu_0,\nu_1,\nu_2,\dots)$. Mention that $\nu_0,\nu_1,\nu_2,\dots$ have a dimension of $3[(M+1)(S_1+1)\quad[S_2 m+S_2 n_2 m]+3(M+1)S_1 S_2 n_1 n_2 m+(M+1)S_1 S_2 n_1 m+(M+1)S_1 S_2 m+(M+1)S_1 S_2 n_2 m]$ and the vector ν satisfies

$$\nu Q=0 \text{ and } \nu e=1.\#(4)$$

Additionally, after the stability needed of the model is met, the stationary probability vector ν can be obtained by applying the following equation:

$$\nu_i=\nu_0 R^i, i\geq 1,\#(5)$$

Second-degree matrix equation

$$R^2 A_2+RA_1+A_0=0\#(6)$$

is met by the [26] least non-negative solution R. The rate matrix is obtained from the matrix quadratic equation. The requirement is satisfied by the rate matrix R, whose order is provided by $3[(M+1)(S_1+1)[S_2 m+S_2 n_2 m]+3(M+1)S_1 S_2 n_1 n_2 m+(M+1)S_1 S_2 n_1 m+(M+1)S_1 S_2 m+(M+1)S_1 S_2 n_2 m]$.

$$RA_2 e=A_0 e\#(7)$$

the sub vector ν_0 can be found by solving the subsequent equation

$$\nu_0(B_{00}+RA_2)=0\#(8)$$

the normalizing condition is subject to

$$\nu_0 e_{3((M+1)(S_1+1)[S_2 m+S_2 n_2 m]+3(M+1)S_1 S_2 n_1 n_2 m+(M+1)S_1 S_2 n_1 m+(M+1)S_1 S_2 m+(M+1)S_1 S_2 n_2 m)=1.0.0}\#(9)$$

Before solving the set of equations mentioned above, we first need to calculate the rate matrix R. This can be done using the Logarithmic Reduction Approach [27], which simplifies the process of finding R .

5. ANALYSIS OF ACTIVE PERIOD

- The time frame that begins when users enter the empty system and concludes when the system size first falls to zero is known as the busy period. Therefore, the first phase of the shift from level 1 to level 0 is the analogue of the busy period. When a state at any other level is visited at least once, the busy cycle is the time it takes to return to level zero.

- Let's use the concept of the basic time to examine the busy time. The QBD process is nothing more than the initial transition period between the levels k_1 to $k_1 - 1$, where $k_1 \geq 1$.
- The boundary states, or the circumstances when $k_1 = 0$, require a distinct discussion. For any level k_1 where $k_1 \geq 1$, it is evident that there are $3[(M+1)(S_1+1)[S_2 m + S_2 n_2 m] + 3(M+1)S_1 S_2 n_1 n_2 m + (M+1)S_1 S_2 n_1 m + (M+1)S_1 S_2 m + (M+1)S_1 S_2 n_2 m]$ states. For this reason, the h^{th} state of level k_1 may be denoted as (k_1, h) when each state is placed in lexicographic order.
- Let $H_{hh'}(k_1, x)$ represent the probability with the condition that, subject to the restriction that it begin in the state (k_1, h) at time $t = 0$, the QBD process visits level $k_1 - 1$ to modify k_1 transitions to the left and also enters the state (k_1, h'). Let us present the idea of the joint transform.

$$\bar{H}_{hh'}(z,s) = \sum_{k_1=1}^{\infty} z_1^k \int_0^{\infty} e^{-sx} dH_{hh'}(k_1, x); |z| \leq 1, \mathrm{Re}(s) \geq 0\#(10)$$

and the matrix is denoted as $\bar{H}(z,s) = \bar{H}hh'(z,s)$. The following equations can be easily satisfied by the matrix $\bar{H}(z,s)$:

$$\bar{H}(z,s) = z[sI - A_1]^{-1}A_2 + [sI - A_1]^{-1}A_0\bar{H}^2(z,s)\#(11)$$

- Without taking into account the boundary states, the initial travel time should be computed with the matrix $H = H_{hh'} = \bar{H}(1,0)$ after evaluating the rate matrix R, we can quickly get the matrix H by applying the result

$$H = -[A_1 + RA_2]^{-1}A2\#(12)$$

- In the absence of such, it is possible to calculate the H matrix values by applying the idea of a logarithmic reduction process. $\bar{H}^{0,0}(1,0)$ provides the following equation, which is related to boundary level zero.

$$\bar{H}^{0,0}(z,s) = [sI - H_{00}]^{-1}A_0\bar{H}(z,s).\#(13)$$

- The next moments are simple to evaluate because the three matrices namely, $\mathrm{H}, \bar{H}^{(1,0)}$ and $\bar{H}^{(0,0)}(1,0)$ are stochastic. At $s = 0, z = 1$,

$$\vec{J_1} = -\frac{\partial}{\partial s}\bar{H}(z,s) = -[A_1 + A_0(H+I)]^{-1}e\#(14)$$

$$\vec{J_2} = \frac{\partial}{\partial s}\bar{H}(z,s) = -[A_1 + A_0(H+I)]^{-1}A_2 e\#(15)$$

$$\vec{J_1}^{(0,0)} = -\frac{\partial}{\partial z}\bar{H}^{(0,0)}(z,s) = -H_{00}^{-1}[e + A_0\vec{J_1}]\#(16)$$

$$\vec{J_2}^{(0,0)} = \frac{\partial}{\partial z}\bar{H}^{(0,0)}(z,s) = -H_{00}^{-1}[A_0\vec{J_2}]\#(17)$$

6. PERFORMANCE MEASURES

We examine our model's qualitative behavior under steady conditions. Let $\mathrm{x}(d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9)$ stand for the probability of the stable state, where the quantity of clients in the orbit I is d_1, the quantity of customers in the orbit II is d_2, server status I is d_3, server status II is d_4, d_5, d_6, d_7, d_8 and d_9 represent the number of commodity I, commodity II, service phase I, service phase II and arrival phases, respectively.

- Probability that server-1 is idle and server-2 is idle:

$$P_{S1IS2I} = \sum_{d_5=0}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_9=1}^{m} x_{0000d_5d_6d_9}$$

- Probability that server-1 is idle and server-2 is busy with re-service:

$$P_{S1IS2BR} = \sum_{d_5=0}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_8=1}^{n_2} \sum_{d_9=1}^{m} x_{0d_201d_5d_6d_8d_9}$$

- Chance of the server that server- 1 is in close down and server- 2 is in idle:

$$P_{S1CS2I} = \sum_{d_5=1}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_9=1}^{m} x_{0030d_5d_6d_9}$$

- chance of the server- 1 is in close down and server- 2 is busy with re-service:

$$P_{S1CS2BR} = \sum_{d_2=1}^{M} \sum_{d_5=1}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_8=1}^{n_2} \sum_{d_9=1}^{m} x_{0d_231d_5d_6d_8d_9}$$

- Probability that server-1 is idle in working vacation and server-2 is idle:

$$P_{S1IWVS2I} = \sum_{d_5=0}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_9=1}^{m} x_{0040d_6d_5d_9}$$

- Probability that server-1 is idle in working vacation and server-2 is busy with re-service:

$$P_{S1IWVS2BR} = \sum_{d_2=1}^{M} \sum_{d_5=0}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_8=1}^{n_2} \sum_{d_9=1}^{m} x_{0d_241d_5d_6d_8d_9}$$

- Probability that server- 1 is busy in normal service and server- 2 is idle:

$$P_{S1BNSS2I} = \sum_{d_1=1}^{\infty} \sum_{d_5=1}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_7=1}^{n_1} \sum_{d_9=1}^{m} x_{d_1010d_5d_6d_7d_9}$$

- Probability that server-1 is busy in normal service and server-2 is busy in re-service:

$$P_{S1BNSS2BR} = \sum_{d_1=1}^{\infty} \sum_{d_2=1}^{M} \sum_{d_5=1}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_7=1}^{n_1} \sum_{d_8=1}^{n_2} \sum_{d_9=1}^{m} x_{d_1d_211d_5d_6d_7d_8d_9}$$

- Probability that server-1 is in repair process and server-2 is in idle:

$$P_{S1RS2I} = \sum_{d_1=1}^{\infty} \sum_{d_5=1}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_9=1}^{m} x_{d_1020d_5d_6d_9d_{10}}$$

- Probability that server- 1 is in repair process and server- 2 is busy with re-service:

$$P_{S1RS2BR} = \sum_{d_1=1}^{\infty} \sum_{d_2=1}^{M} \sum_{d_5=1}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_8=1}^{n_2} \sum_{d_9=1}^{m} x_{d_1d_2}21d_5d_6d_8d_9$$

- Probability that server- 1 is busy in working vacation and server- 2 is in idle:

$$P_{S1BWVS2I} = \sum_{d_1=1}^{\infty} \sum_{d_5=1}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_7=1}^{n_1} \sum_{d_9=1}^{m} x_{d_1050d_5d_6d_7d_9}$$

- Probability that server-1 is busy in working vacation and server-2 is busy with reservice:

$$P_{S1BWVS2R} = \sum_{d_1=1}^{\infty} \sum_{d_2=1}^{M} \sum_{d_5=1}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_7=1}^{n_1} \sum_{d_8=1}^{n_2} \sum_{d_9=1}^{m} x_{d_1 d_2 5 1 d_5 d_6 d_7 d_8 d_9}$$

- Probability that both the servers are in busy mode:

$$P_{S1BNMS2R} = \sum_{d_1=1}^{\infty} \sum_{d_2=1}^{M} \sum_{d_5=1}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_7=1}^{n_1} \sum_{d_8=1}^{n_2} \sum_{d_9=1}^{m} x_{d_1 d_2 1 1 d_5 d_6 d_7 d_8 d_9}$$

- Expected number of customers in orbit I:

$$E[N_1] = \sum_{d_1=0}^{\infty} d_1 x_{d_1} e$$

- Expected number of customers in orbit II:

$$E[N_2] = \sum_{d_1=1}^{\infty} \sum_{d_2=1}^{M} \sum_{d_3=0}^{5} \sum_{d_4=0}^{1} d_2 x_{d_1 d_2 d_3 d_4} e$$

- Expected number of customers in the system:

$$E_{\text{system}} = E[N_1] + E[N_2] + P_{\text{busy}}$$

- Expected retrial rate:

$$E_{RR} = (\sigma_1 + \sigma_2) \sum_{d_1=1}^{\infty} \sum_{d_2=1}^{M} \sum_{d_5=1}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_9=1}^{m} x_{d_1 d_2 0 0 d_5 d_6 d_9}$$

$$+\sigma_1 \sum_{d_1=1}^{\infty} \sum_{d_2=0}^{M} \sum_{d_5=1}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_9=1}^{m} x_{d_1 d_2 4 1 d_5 d_6 d_9}$$

$$+\sigma_2 \sum_{d_1=0}^{\infty} \sum_{d_2=1}^{M} \sum_{d_5=1}^{S_1} \sum_{d_6=1}^{S_2} \sum_{d_9=1}^{m} x_{d_1 d_2 1 0 d_5 d_6 d_9}$$

- Probability that server-1 is idle:

$$P_{idle} = P_{1NM} + P_{IWV}$$

- Probability that server-1 is busy:

$$P_{busy} = P_{BNM} + P_{BWV}$$

7. NUMERICAL IMPLEMENTATION

In this section, we examine the output of our system using graphical and numerical representations. With a mean value of 1, these arrival processes $ERL-A, EXP-A$ and $HEX-A$ have zero correlation and it correspond to the renewal process. Erlang Arrival (ERL-A):

$D_0 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$

Exponential Arrival(EXP-A):

$D_0 = [-1], D_1 = [1]$

Hyper exponential Arrival(HEX-A):

$D_0 = \begin{bmatrix} -1.90 & 0 \\ 0 & -0.19 \end{bmatrix}, D_1 = \begin{bmatrix} 1.710 & 0.190 \\ 0.171 & 0.019 \end{bmatrix}$

Consider the following PH distributions for the service progression:
Erlang Service(ERL-S):

$\gamma_1 = \gamma_2 = [1,0];\ U_1 = U_2 = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$

Exponential Service(EXP-S):

$\gamma_1 = \gamma_2 = 1;\ U_1 = U_2 = [-1]$

Hyper Exponential Service(HEX-S):

$\gamma_1 = \gamma_2 = [0.8,0.2];\ U_1 = U_2 = \begin{bmatrix} -2.80 & 0 \\ 0 & -0.28 \end{bmatrix}$

7.1. Illustration 1

We investigate how the retrial rate (σ_1) affects the ($E_{\text{Orbit}\, I}$) in tables 1 to 3. Fix $\lambda = 1$, $\mu_1 = 8, \mu_2 = 11, \sigma_2 = 15, \tau = 3, \delta = 1, \beta = 2, M = 2, p = 0.5, q = 0.5, f_1 = 0.5, f_2 = 0.5,$ $S_2 = 5, S_1 = 6, s_1 = 3, \eta = 2$ and $\theta = 0.8$ such that the system remains stable. The following analysis is based on the observations from Table 1 to Table 3 : When we increase the retrial rate while accounting for both arrival and service periods, the E_{OrbitI} decreases. From the perspective of service and arrival periods, $E_{\text{Orbit}\, I}$ decreases significantly for HEX-S and gradually for $ERL - S$ when the retrial rate increases; in contrast to EXP-S, E_{OrbitI} decreases significantly in $ERL - S$ and slowly in HEX-S when retrial times are taken into consideration.

Table 1: *Retrial rate (σ_1) vs E_{OrbitI} -ERL-A*

σ_1	ERL-S	EXP-S	HEX-S
13	1.750777248	1.739976801	1.731201809
14	1.746243381	1.735298055	1.726225649
15	1.742263775	1.731198637	1.721877722
16	1.738738655	1.727574398	1.718044536
17	1.735591201	1.724344979	1.714638486
18	1.732761275	1.721447368	1.711590862
19	1.730201106	1.718831443	1.708847012
20	1.727872243	1.716456823	1.70636292
21	1.72574337	1.714290601	1.704102731
22	1.723788709	1.712305676	1.702036943

Table 2: *Retrial rate (σ_1) vs E_{OrbitI} -EXP-A*

σ_1	ERL-S	EXP-S	HEX-S
13	1.798046389	1.787136458	1.784617201
14	1.792961497	1.781868423	1.778952472
15	1.788508844	1.777264719	1.774013311
16	1.78457188	1.773203235	1.769671376

17	1.781061513	1.769592361	1.76582149
18	1.777908448	1.766353113	1.762383101
19	1.775057948	1.763433734	1.759292408
20	1.772466178	1.760785903	1.756498218
21	1.770097598	1.758372024	1.753958974
22	1.767923067	1.756161254	1.751640592

***Table 3**: Retrial rate(σ_1) vs E_{OrbitI} - HEX-A*

σ_1	$ERL-S$	$EXP-S$	$HEX-S$
13	2.092643438	2.06382038	2.040366951
14	2.085895278	2.056691266	2.032341245
14	2.079974135	2.050458089	2.025356525
15	2.074724232	2.044952872	2.019217965
16	2.070027656	2.040047824	2.013776693
17	2.065793473	2.035643975	2.008917002
18	2.065793473	2.035643975	2.008917002
19	2.06195037	2.031663454	2.004547612
20	2.058441583	2.02804412	2.000595542
21	2.055221318	2.024735753	1.997001741

7.2. Illustration 2

The effects of the vacation rate (η) in relation to P_{busy} have been examined in figures 2 to 4 . Fix $\lambda = 1, \mu_1 = 8, \mu_2 = 11, \sigma_2 = 15, \tau = 3, \delta = 1, \beta = 2, M = 2, p = 0.5, q = 0.5, f_1 = 0.5, f_2 = 0.5, S_2 = 5, S_1 = 6, s_1 = 3, \sigma_1 = 13$ and $\theta = 0.8$ such that the system remains stable. We note that from Figure 2 to Figure 4, as follows:

For every combination of arrival-time and service-time distributions, it is found that the chance that the server is busy, P_{busy}, decreases as the vacation rate rises. Because a larger vacation rate results in the server spending more time away from active service, which lowers the proportion of time it remains busy, the phenomenon is constant.

7.3. Illustration 3

To examine the effect on the E_{system} in the corresponding Figures 5 to 13 . Fix $\lambda = 1$, $\mu_2 = 11, \sigma_2 = 15, \delta = 1, \beta = 2, M = 2, p = 0.5, q = 0.5, f_1 = 0.5, f_2 = 0.5, S_2 = 5, S_1 = 6,$ $s_1 = 3, \sigma_1 = 13$ and $\theta = 0.8$ such that the system remains stable.

The decrease in E_{system} is particularly noticeable in the $HEX-S$, when server 1's service rate rises. In comparison, the $ERL-S$ (Erlang service-time) instance shows a much slower decline. This shows that when service capacity increases, the system becomes much more efficient under $HEX-S$, although the benefit is less noticeable when service times have an Erlang distribution.

Analyzing the impact of higher repair rates under various arrival distributions reveals that $ERL - A$ (Erlang arrival time) only slightly decreases. However, for $EXP - A$ (exponential arrival time), the decline is much more pronounced. This implies that systems with exponential arrivals gain from increases in repair efficiency more significantly than systems with Erlang arrivals.

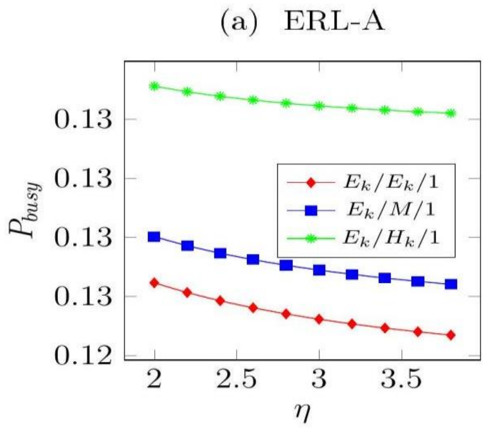

Figure 2: Vacation rate vs P_{busy}.

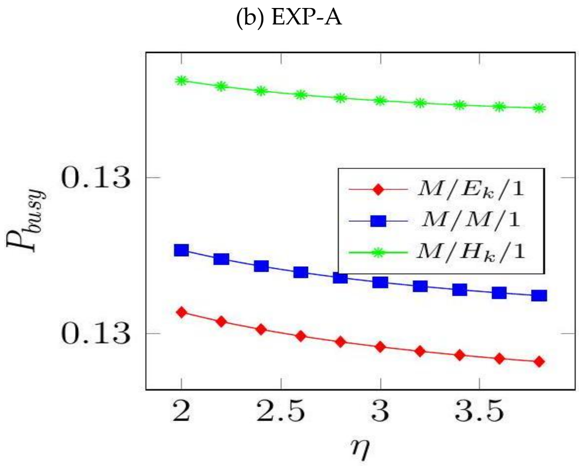

Figure 3: Vacation rate vs P_{busy}.

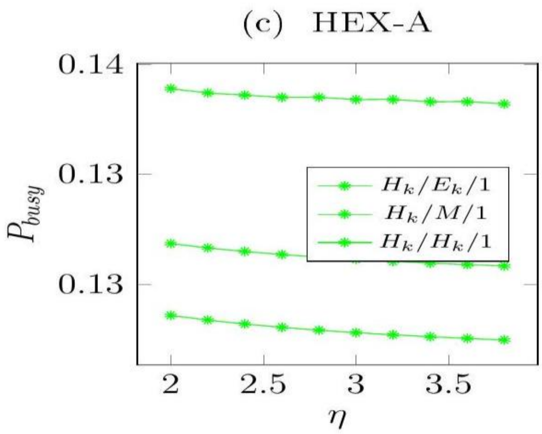

Figure 4: Vacation rate vs P_{busy}.

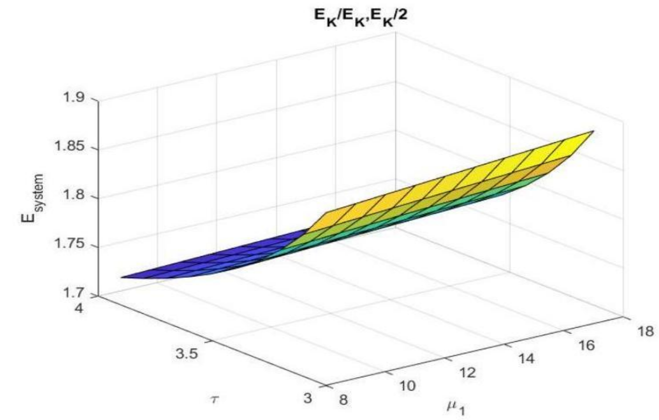

Figure 5: Server-1 service rate and Repair Rate vs E_{system} .

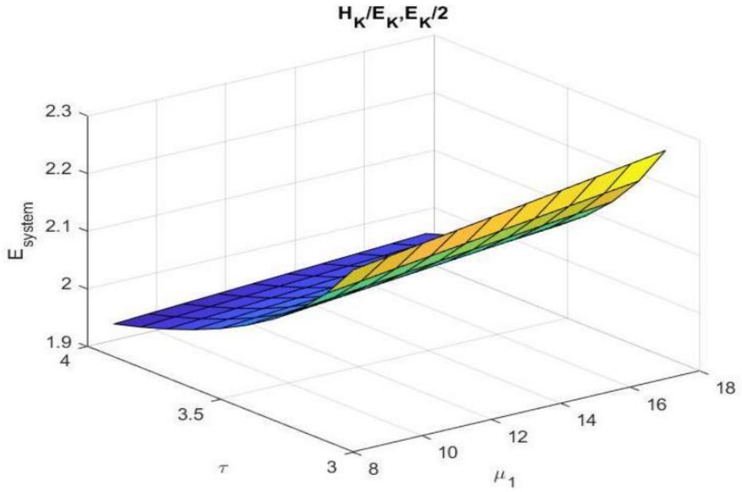

Figure 7: Server-1 service rate and Repair Rate vs E_{system} .

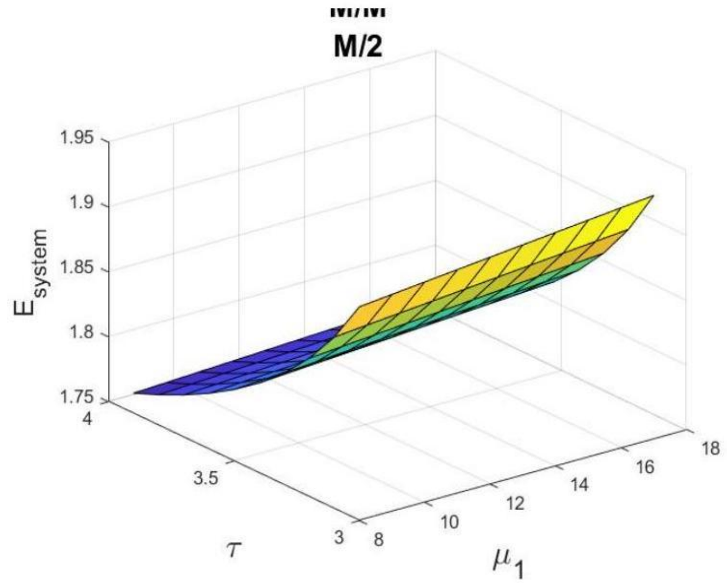

Figure 9: Server-1 service rate and Repair Rate vs E_{system} .

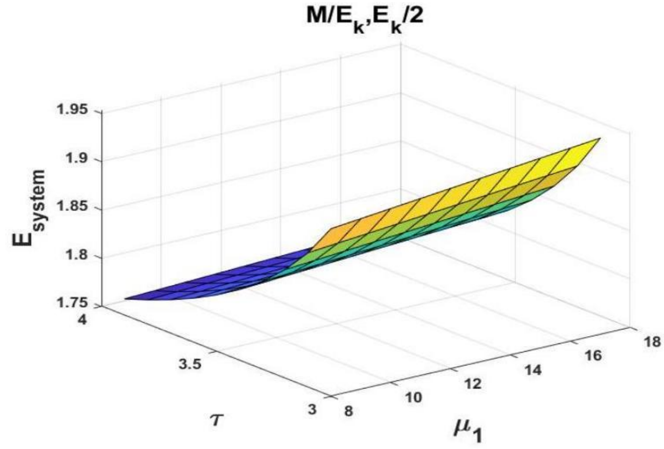

Figure 6: Server-1 service rate and Repair Rate vs E_{system} .

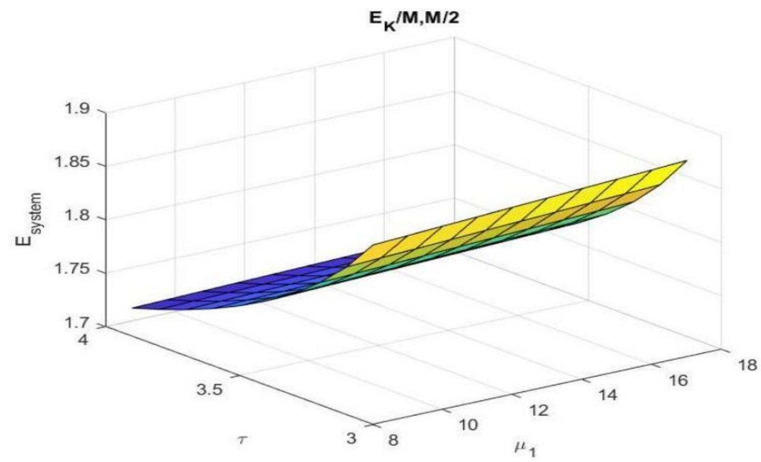

Figure 8: Server-1 service rate and Repair Rate vs E_{system} .

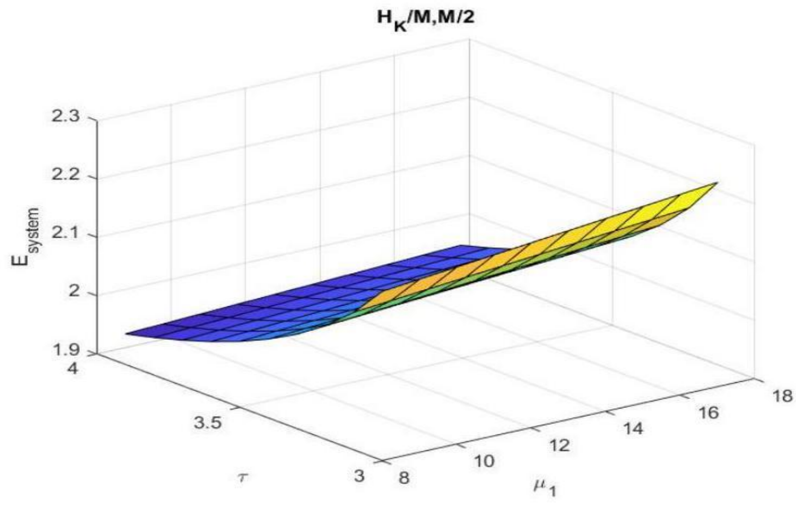

Figure 10: Server-1 service rate and Repair Rate vs E_{system} .

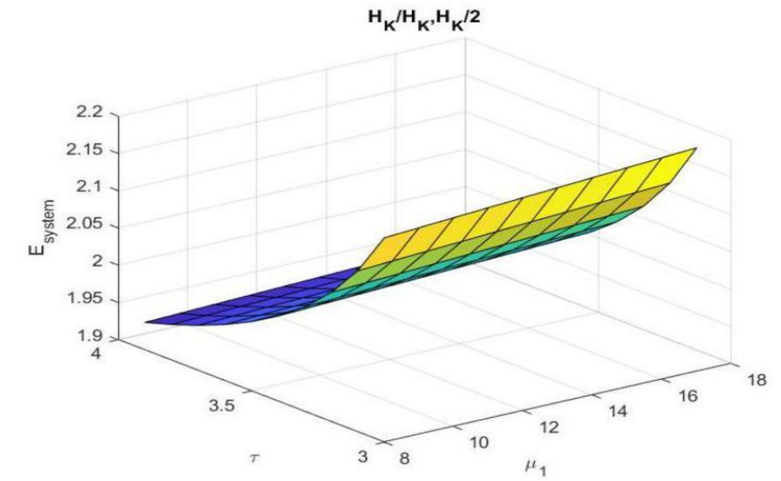

Figure 11: Server-1 service rate and Repair Rate vs E_{system}.

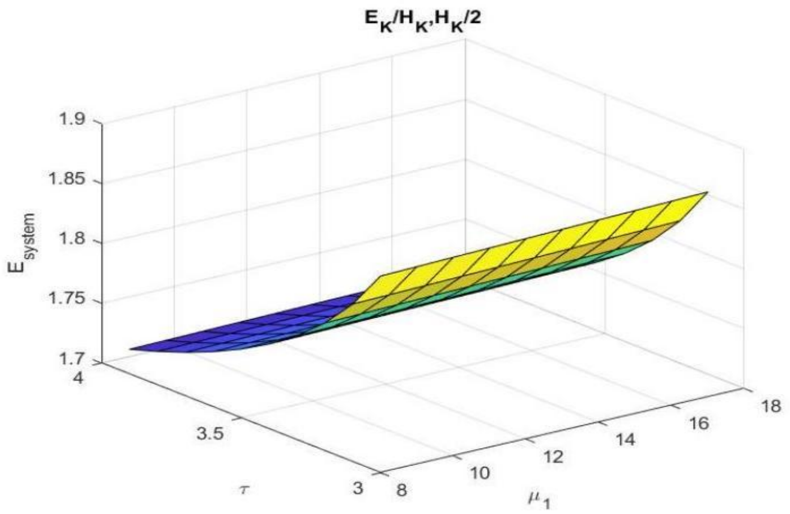

Figure 12: Server-1 service rate and Repair Rate vs E_{system}.

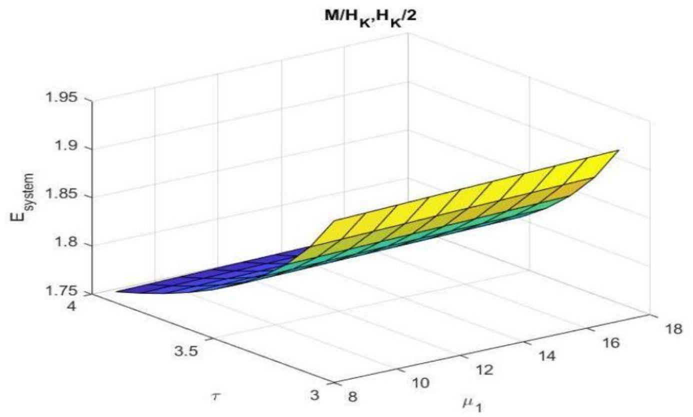

Figure 13: Server-1 service rate and Repair Rate vs E_{system}.

4. CONCLUSION

A two-server retrial queueing-inventory system with double orbit, working vacation, breakdown, repair, re-service, emergency replenishment, and (s, S) policy was examined. Customers arrive in accordance with a Markovian arrival process and a phase-type distribution adapts the service times. A continuous-time Markov chain with a discrete state space is used to create the system's mathematical model. The matrix-analytic method is used to derive the stability condition and subsequently the steady-state distribution. Numerical examples are used to demonstrate how the parameters affect the performance measures. Considering the QIS model with a batch Markov process (BMAP) and perishable inventory could be an extension of this work.

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