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ON SOME HYPERSINGULAR INTEGRALS

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ABSTRACT

The paper examines different types of hypersingular integrals with the Cauchy kernel on a segment and a unit circle and defines them using specific methods. It presents more general definitions for one-dimensional hypersingular integrals with the Cauchy kernel based on Hadamard's integral in the sense of a finite part. The paper also establishes the existence theorems of these hypersingular integrals and ***formulas, which*** *demonstrates the accuracy of the resulting integrals that are applied in various applications and engineering problem-solving.* ***The proposed formulas are straightforward to calculate, making the new approximate method reliable and easy to apply and the obtained*** *numerical results demonstrate* ***the stability and efficiency of the approach.***

BƏZİ HİPERSİNQULYAR İNTEQRALLAR HAQQINDA

XÜLASƏ

İşdə parçada və vahid çevrədə müxtəlif növ Koşi nüvəli hipersinqulyar inteqrallar öyrənilir və onlar xüsusi üsulla təyin edilir, belə ki, bu birölçülü Koşi nüvəli hipersinqulyar inteqrallara J.Adamar mənada inteqral anlayışından istifadə edərək uyğun olaraq müxtəlif şəkildə təriflər verilir.İşdə həmçinin verilmiş hipersinqulyar inteqralların varlığı haqqında teoremlər isbat olunur və bu inteqrallar üçün mühəndislik məsələlərində və müxtəlif sahələrdə geniş tətbiq edilə bilən düsturlar verilir.

Açar sözlər: hipersinqulyar inteqral, approksimasiya operatorları, yığılma sürəti, Koşi nüvəsi.

О НЕКОТОРЫХ ГИПЕРСИНГУЛЯРНЫХ ИНТЕГРАЛАХ

АБСТРАКТ

В работе изучаются разные виды гиперсингулярных интегралов с ядром Коши на отрезке и на единичной окружности и определяются они специальным образом. Для одномерных гиперсингулярных интегралов с ядром Коши даются более общие, чем традиционные, определения, использующие идею Адамара о понятии интеграла в смысле конечной части. В работе также доказываются теоремы о существовании этих гиперсингулярных интегралов и показывается справедливость формул для интегралов, дающих более точные результаты в приложениях в разных областях и при решении различных инженерных задач.

Ключевые слова: Гиперсингулярный интеграл, аппроксимирующие операторы, скорости сходимости, ядро Коши.

1.Introduction

An active development of numerical methods for solving hypersingular integral equations is of considerable interest in modern numerical analysis. This is due to the fact that hypersingular integral equations have numerous applications in acoustics, aerodynamics, fluid mechanics, electrodynamics, elasticity, fracture mechanics, geophysics and etc. [8-12,13-21,27,29-30] Therefore the construction and justification of numerical schemes for approximate solutions of hypersingular integral equations is a topical issue and numerous works are devoted to their development.The development of constructive methods for solving hypersingular integral equations is impossible without studying the properties of hypersingular integral operators contained in these equations, and is associated with the approximation of such operators, which indicates the actuality of the subject of our mauscript. Hypersingular integrals were introduced by J. Hadamard for the solution of the Cauchy problem for a linear partial differential equations of a hyperbolic type. They also arise in solving Neumann problem for the Laplace equation, in solving integral equations of the linear theory of a bearing surface, in inverting generalized Riesz potentials, when presenting some classes of pseudo-differential operators and in other areas of mathematics and mechanics.

Present paper consist of introduction, two chapters and references list.

In the first chapter is investigated hypersingular integrals of the following forms

$$\int_a^b \frac{g(x)}{(x-x_0)^2}dx\ , \quad x_0 \in (a,b)$$

$$\int_a^b \frac{g(x)}{(x-x_0)^m}dx\ , \quad m \ge 3, \quad m \in N\ , \quad x_0 \in (a,b),$$

where the function $g(x)$ is Lebesgue integrable on the interval $[a,b]$.

In the second chapter are considered hypersingular integrals of the following forms

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau-t)^2}d\tau, \quad t \in \gamma_0$$

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau-t)^m}d\tau\ \ , \ m \ge 3, \ m \in N\ , \ t \in \gamma_0\ ,$$

where the function $\varphi(t)$ is Lebesgue integrable on the unit circle $\gamma_0 = \{t \in C : |t| = 1\}$.

In both chapters hypersingular integral is defined by the special way, using using the idea of Hadamard [28] finite part integral and are proved theorems about existence of given hypersingular integrals.

Note that, for the singular integral operators with Cauchy kernel and Hilbert kernel similar approximations and their applications to the singular integral equations are given in [1-5,22-26].

2. Cauchy hypersingular integrals on interval.

Consider the integral

$$\int_a^b \frac{g(x)}{(x-x_0)^2}dx\,,\quad x_0\in(a,b), \tag{1}$$

where the function $g(x)$ is Lebesgue integrable on the interval $[a,b]$.

If we define this integral in a similar manner to the Cauchy integral, even $g\equiv 1$ we get the divergent integral:

$$\lim_{\varepsilon\to 0+}\left(\int_a^{x_0-\varepsilon}\frac{1}{(x-x_0)^2}dx+\int_{x_0+\varepsilon}^{b}\frac{1}{(x-x_0)^2}dx\right)=\lim_{\varepsilon\to 0+}\left(\frac{2}{\varepsilon}-\frac{1}{x_0-a}+\frac{1}{x_0-b}\right)=\infty\,.$$

Therefore, using the idea of Hadamard finite part integral [28], we will define the integral (1) as follows.

Definition 1. If a finite limit

$$\lim_{\varepsilon\to 0+}\left(\int_a^{x_0-\varepsilon}\frac{g(x)}{(x-x_0)^2}dx+\int_{x_0+\varepsilon}^{b}\frac{g(x)}{(x-x_0)^2}dx-\frac{2g(x_0)}{\varepsilon}\right),$$

exists, then the value of this limit is referred to as the hypersingular integral of the function $\frac{g(x)}{(x-x_0)^2}$ on $[a,b]$ and is denoted by $\int_a^b\frac{g(x)}{(x-x_0)^2}dx$.

The hypersingular integral (1) was studied in [7]. In [7] it is proved that, if $g(x)\in H_1(\alpha)$, i.e. $g(x)$ is differentiable on the interval $[a,b]$and $g'(x)$includes to the class of the Hölder continuous functions with exponent α , (i.e. the class of the functions that satisfy the following condition $\exists M_0>0\quad \forall t_1,t_2\in R:\ \left|g'(x_1)-g'(x_2)\right|\le M_0\cdot\left|x_1-x_2\right|^{\alpha}$) then (1) exists and the following equality holds:

$$\int_a^b\frac{g(x)}{(x-x_0)^2}dx=\frac{g(b)}{x_0-b}-\frac{g(a)}{x_0-a}-\int_a^b\frac{g'(x)}{x_0-x}dx\,, \tag{2}$$

where the integral standing on the right-hand side is understood in the sense of the Cauchy principal value.

Now consider the integral [6]

$$\int_a^b\frac{g(x)}{(x-x_0)^m}dx\,,\quad m\ge 3,\quad m\in N\,,\quad x_0\in(a,b), \tag{3}$$

where the function $g(x)$ is Lebesgue integrable on $[a,b]$.

As in the case of the hypersingular integral (1) if we define the integral (3) in a similar manner to the Cauchy integral, even $g \equiv 1$ we get the divergent integral. Therefore, using the idea of Hadamard finite part integral [28], we will define the integral (3) as follows.

Definition 2. Let $m \geq 3,\ m \in N$, the function $g(x)$ is Lebesgue integrable on the interval $[a,b]$ and is differentiable $(m-2)$ times at the point $x_0 \in (a,b)$.

If a finite limit

$$\lim_{\varepsilon\to 0+}\left(\int_a^{x_0-\varepsilon}\frac{g(x)}{(x-x_0)^m}dx+\int_{x_0+\varepsilon}^{b}\frac{g(x)}{(x-x_0)^m}dx-2\sum_{k=0}^{p-1}\frac{g^{(2k)}(x_0)}{(2k)!(2p-2k-1)\varepsilon^{2p-2k-1}}\right)$$

when $m = 2p$, $p \in N$,

$$\lim_{\varepsilon\to 0+}\left(\int_a^{x_0-\varepsilon}\frac{g(x)}{(x-x_0)^m}dx+\int_{x_0+\varepsilon}^{b}\frac{g(x)}{(x-x_0)^m}dx-2\sum_{k=0}^{p-1}\frac{g^{(2k+1)}(x_0)}{(2k+1)!(2p-2k-1)\varepsilon^{2p-2k-1}}\right)$$

when $m = 2p+1,\ p \in N$,

exists, then the value of this limit is referred to as the hypersingular integral of the function $\frac{g(x)}{(x-x_0)^m}$ on $[a,b]$ and is denoted by $\int_a^b \frac{g(x)}{(x-x_0)^m}dx$.

From definitions 1 and 2 it follows that, if the function $g(x)$ is differentiable on $[a,b]$ and the integral $\int_a^b \frac{g'(x)}{(x-x_0)^{m-1}}dx$ exists, then (3) exists also, and the following formula by parts holds:

$$\int_a^b \frac{g(x)}{(x-x_0)^m}dx=\frac{1}{m-1}\left[\frac{g(a)}{(a-x_0)^{m-1}}-\frac{g(b)}{(b-x_0)^{m-1}}+\int_a^b\frac{g'(x)}{(x-x_0)^{m-1}}dx\right]. \qquad (4)$$

From (4) follows that, if the function $g(x)$ is differentiable $(m-2)$ times on $[a,b]$ and $(m-2)^{\text{th}}$ derivative of the function $g^{(m-2)}(x)$ absolutely continuous on $[a,b]$, then the integral (3) exists almost everywhere for all $x_0 \in (a,b)$ and the following equality is true:

$$\int_a^b \frac{g(x)}{(x-x_0)^m}dx=\sum_{k=0}^{m-2}\frac{(m-2-k)!}{(m-1)!}\left[\frac{g^{(k)}(a)}{(a-x_0)^{m-1-k}}-\frac{g^{(k)}(b)}{(b-x_0)^{m-1-k}}\right]+\frac{1}{(m-1)!}\int_a^b\frac{g^{(m-1)}(x)}{x-x_0}dx$$

If we apply formula (4) to the integral $\int_a^b \frac{u(x)v(x)}{(x-x_0)^{m+1}}dx$, we derive that, if the functions $u(x)$ and $v(x)$ have absolutely continuous $(m-2)^{\text{th}}$ derivatives on $[a,b]$, then the following relation holds almost everywhere for all $x_0 \in (a,b)$

$$\int_a^b \frac{u(x)v(x)}{(x-x_0)^{m+1}}dx=\frac{1}{m}\left[\frac{u(a)v(a)}{(a-x_0)^m}-\frac{u(b)v(b)}{(b-x_0)^m}+\int_a^b\frac{u'(x)v(x)+u(x)v'(x)}{(x-x_0)^m}dx\right].$$

As result we get the following integration by parts formula:

$$\int_a^b \frac{u(x)}{(x-x_0)^m}\,dv(x) = \left.\frac{u(x)v(x)}{(x-x_0)^m}\right|_a^b - \int_a^b v(x)\,d\left(\frac{u(x)}{(x-x_0)^m}\right) =$$

$$= \frac{u(b)v(b)}{(b-x_0)^m} - \frac{u(a)v(a)}{(a-x_0)^m} - \int_a^b \frac{u'(x)(x-x_0)-mu(x)}{(x-x_0)^{m+1}}v(x)\,dx. \qquad (5)$$

3. Cauchy hypersingular integrals on unit circle.

Let's consider the integral [3]

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau-t)^2}\,d\tau, \quad t\in\gamma_0 , \qquad (6)$$

where the function $\varphi(t)$ is Lebesgue integrable on $\gamma_0 = \{t \in C : |t| = 1\}$.

Using definition1 for the hypersingular integral on interval, define the integral (6) in the following form

Definition 3. If a finite limit

$$\lim_{\varepsilon\to 0+}\left(\int_{\gamma_\varepsilon} \frac{\varphi(\tau)}{(\tau-t)^2}\,d\tau + \frac{2i\varphi(t)}{\varepsilon\cdot t}\right),$$

exists, then the value of this limit is called the hypersingular integral of the function $\frac{\varphi(\tau)}{(\tau-t)^2}$, and is denoted by $\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau-t)^2}\,d\tau$, where $\gamma_\varepsilon = \{\tau \in \gamma_0 : |\tau - t| > \varepsilon\}$.

From Definitions 1 and 3 we can deduce that, if $t = e^{ix_0}$, $x_0 \in (-\pi,\pi)$, then

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau-t)^2}\,d\tau = \lim_{\varepsilon\to 0+}\left(\int_{\gamma_\varepsilon} \frac{\varphi(\tau)}{(\tau-t)^2}\,d\tau + \frac{2i\varphi(t)}{\varepsilon\cdot t}\right) = \lim_{\varepsilon\to 0+}\left(\int_{x_0+\delta(\varepsilon)}^{x_0+2\pi-\delta(\varepsilon)} \frac{\varphi\left(e^{ix}\right)\cdot ie^{ix}}{\left(e^{ix}-e^{ix_0}\right)^2}\,dx + \frac{2i\varphi\left(e^{ix_0}\right)}{\varepsilon\cdot e^{ix_0}}\right) =$$

$$= \lim_{\varepsilon\to 0+}\left(\int_{[-\pi,\pi]\setminus(x_0-\delta(\varepsilon),x_0+\delta(\varepsilon))} \frac{\varphi\left(e^{ix}\right)\cdot ie^{ix}}{(x-x_0)^2}\cdot\left(\frac{x-x_0}{e^{ix}-e^{ix_0}}\right)^2 dx + \frac{2i\varphi\left(e^{ix_0}\right)}{\varepsilon\cdot e^{ix_0}}\right) =$$

$$= \int_{-\pi}^{\pi} \frac{\varphi\left(e^{ix}\right)\cdot ie^{ix}}{\left(e^{ix}-e^{ix_0}\right)^2}\,dx + \frac{2i\varphi\left(e^{ix_0}\right)}{e^{ix_0}}\cdot \lim_{\varepsilon\to 0+}\left(\frac{1}{\varepsilon} - \frac{1}{\delta(\varepsilon)}\right), \qquad (7)$$

where $\delta(\varepsilon) = 2\arcsin\frac{\varepsilon}{2}$. Since

$$\lim_{\varepsilon\to 0+}\left(\frac{1}{\varepsilon}-\frac{1}{\delta(\varepsilon)}\right)=\lim_{\varepsilon\to 0+}\left(\frac{1}{\varepsilon}-\frac{1}{2\arcsin\frac{\varepsilon}{2}}\right)=0,$$

then from (7) it follows that,

$$\int_{\gamma_0}\frac{\varphi(\tau)}{(\tau-t)^2}d\tau=\int_{-\pi}^{\pi}\frac{\varphi\left(e^{ix}\right)\cdot ie^{ix}}{\left(e^{ix}-e^{ix_0}\right)^2}dx. \tag{8}$$

The formula (8) shows that, by means of the substitution $t=e^{ix}$ hypersingular integral (6) is reduced to the hypersingular integral on an interval.

Let's calculate the following integral

$$\int_{\gamma_0}\frac{d\tau}{(\tau-t)^2},\ t\in\gamma_0.$$

We get

$$\int_{\gamma_0}\frac{d\tau}{(\tau-t)^2}=\lim_{\varepsilon\to 0+}\left(\int_{\gamma_\varepsilon}\frac{d\tau}{(\tau-t)^2}+\frac{2i}{\varepsilon t}\right)=\lim_{\varepsilon\to 0+}\left(\frac{1}{t-t\cdot e^{-i\delta(\varepsilon)}}-\frac{1}{t-t\cdot e^{i\delta(\varepsilon)}}+\frac{2i}{\varepsilon\cdot t}\right)=$$

$$=\frac{1}{t}\lim_{\varepsilon\to 0+}\left(\frac{e^{i\delta(\varepsilon)}}{e^{i\delta(\varepsilon)}-1}-\frac{1}{1-e^{i\delta(\varepsilon)}}+\frac{2i}{\varepsilon}\right)=\lim_{\varepsilon\to 0+}\left(\frac{e^{i\delta(\varepsilon)}+1}{e^{i\delta(\varepsilon)}-1}+\frac{2i}{\varepsilon}\right), \tag{9}$$

where $\delta(\varepsilon)=2\arcsin\frac{\varepsilon}{2}\sim\varepsilon$ при $\varepsilon\to 0+$. Since

$$\lim_{\varepsilon\to 0+}\left(\frac{e^{i\delta(\varepsilon)}+1}{e^{i\delta(\varepsilon)}-1}+\frac{2i}{\varepsilon}\right)=\lim_{\varepsilon\to 0+}\left(\frac{2}{i\delta(\varepsilon)}+\frac{2i}{\varepsilon}\right)=2i\cdot\lim_{\varepsilon\to 0+}\left(\frac{1}{\varepsilon}-\frac{1}{\delta(\varepsilon)}\right)=0,$$

then from (9) it follows that,

$$\int_{\gamma_0}\frac{d\tau}{(\tau-t)^2}=0. \tag{10}$$

From (10) it follows that, existence of hypersingular integral $\int_{\gamma_0}\frac{\varphi(\tau)}{(\tau-t)^2}d\tau$ is equavivalent to existence of hypersingular integral $\int_{\gamma_0}\frac{\varphi(\tau)-\varphi(\tau)}{(\tau-t)^2}d\tau$ in the sense of the Cauchy principal value, then the following relation holds:

$$\int_{\gamma_0}\frac{\varphi(\tau)}{(\tau-t)^2}d\tau=\int_{\gamma_0}\frac{\varphi(\tau)-\varphi(\tau)}{(\tau-t)^2}d\tau,$$

where the integral standing on the right-hand side is understood in the sense of the Cauchy principal value.

According to (2) and (8), we get that, if $\varphi \in C^{1,\alpha}(\gamma_0)$, i.e. $\varphi(t)$ is differentiable on the unit circle γ_0 and $\varphi'(t)$ includes to the class of the Hölder continuous functions with exponent α , then (6) exists and the following equality holds:

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau - t)^2} d\tau = \int_{\gamma_0} \frac{\varphi'(\tau)}{\tau - t} d\tau \;, \tag{11}$$

where the integral standing on the right-hand side is understood in the sense of the Cauchy principal value.

State the next useful property of the hypersingular integral in the form (6).

Theorem 1. If the function φ absolutely continuous on γ_0 ,then the hypersingular integral (6) exists for almost all $t \in \gamma_0$,and (11) holds.

Proof: The existence of the derivative $\varphi'(\tau)$ of the function φ for almost all $t \in \gamma_0$ and Lebesgue integrablity of the function $\varphi'(\tau)$ on γ_0 ,follows from absolute continuity of the function φ on γ_0 .

For all points $t \in \gamma_0$, which the function $\varphi(t)$ is differentiable, the following equality is true:

$$\lim_{\varepsilon \to 0+} \left(\frac{\varphi\left(t \cdot e^{-i\delta(\varepsilon)}\right)}{t - t \cdot e^{-i\delta(\varepsilon)}} - \frac{\varphi\left(t \cdot e^{i\delta(\varepsilon)}\right)}{t - t \cdot e^{i\delta(\varepsilon)}} + \frac{2i\varphi(t)}{\varepsilon \cdot t} \right) =$$

$$= \lim_{\varepsilon \to 0+} \left[\frac{\varphi(t) + \varphi'(t) \cdot \left(t \cdot e^{-i\delta(\varepsilon)} - t\right) + o(\varepsilon)}{t - t \cdot e^{-i\delta(\varepsilon)}} - \frac{\varphi(t) + \varphi'(t) \cdot \left(t \cdot e^{i\delta(\varepsilon)} - t\right) + o(\varepsilon)}{t - t \cdot e^{i\delta(\varepsilon)}} + \frac{2i\varphi(t)}{\varepsilon \cdot t} \right] =$$

$$= \frac{\varphi(t)}{t} \lim_{\varepsilon \to 0+} \left(\frac{1}{1 - e^{-i\delta(\varepsilon)}} + \frac{1}{1 + e^{i\delta(\varepsilon)}} + \frac{2i}{\varepsilon} \right) = 0,$$

where $\delta(\varepsilon) = 2\arcsin\frac{\varepsilon}{2}$. If $t = e^{ix_0}$, then from the following relation

$$\int_{\gamma_\varepsilon} \frac{\varphi(\tau)}{(\tau - t)^2} d\tau = \int_{x_0 + \delta(\varepsilon)}^{x_0 + 2\pi - \delta(\varepsilon)} \frac{\varphi\left(e^{ix}\right) \cdot ie^{ix}}{\left(e^{ix} - e^{ix_0}\right)^2} dx = -\int_{x_0 + \delta(\varepsilon)}^{x_0 + 2\pi - \delta(\varepsilon)} \varphi\left(e^{ix}\right) d\left(\frac{1}{e^{ix} - e^{ix_0}}\right) =$$

$$= \frac{\varphi\left(e^{i(x_0 + 2\pi - \delta(\varepsilon))}\right)}{e^{ix_0} - e^{i(x_0 + 2\pi - \delta(\varepsilon))}} - \frac{\varphi\left(e^{i(x_0 + \delta(\varepsilon))}\right)}{e^{ix_0} - e^{i(x_0 + \delta(\varepsilon))}} + \int_{x_0 + \delta(\varepsilon)}^{x_0 + 2\pi - \delta(\varepsilon)} \frac{1}{e^{ix} - e^{ix_0}} \cdot ie^{ix} \varphi'\left(e^{ix}\right) dx =$$

$$= \frac{\varphi\left(t \cdot e^{-i\delta(\varepsilon)}\right)}{t - t \cdot e^{-i\delta(\varepsilon)}} - \frac{\varphi\left(t \cdot e^{i\delta(\varepsilon)}\right)}{t - t \cdot e^{i\delta(\varepsilon)}} + \int_{\gamma_\varepsilon} \frac{\varphi'(\tau)}{\tau - t} d\tau$$

it follows that, hypersingular integral (6) exists for almost all $t \in \gamma_0$,and (11) holds and this completes the proof of the theorem1.

As a result we can we can conclude that, even under weak restrictions on the function $\varphi(\tau)$, (11) holds.

Now consider the integrals of the following form:

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau - t)^m} d\tau \quad , \ m \geq 3,\ m \in N,\ t \in \gamma_0 \ , \qquad (12)$$

where the function $\varphi(t)$ is Lebesgue integrable on $\gamma_0 = \{t \in C : |t| = 1\}$.

Using definition 2 for hypersingular integral on interval, define the integral (12) as follows.

Definition 4. Let $m \geq 3$, $m \in N$, the function $\varphi(t)$ is Lebesgue integrable on the unit circle γ_0 and is differentiable $(m-2)$ times at the point $t \in \gamma_0$.

If a finite limit

$$\lim_{\varepsilon \to 0+} \left(\int_{\gamma_\varepsilon} \frac{\varphi(\tau)}{(\tau - t)^m} d\tau - 2i \sum_{k=0}^{p-1} \frac{\left[\varphi\left(e^{ix}\right) e^{ix} \left(\dfrac{x - x_0}{e^{ix} - e^{ix_0}} \right)^m \right]^{(2k)} (x_0)}{(2k)!(2p - 2k - 1)[\delta(\varepsilon)]^{2p-2k-1}} \right)$$

when $m = 2p$, $p \in N$,

$$\lim_{\varepsilon \to 0+} \left(\int_{\gamma_\varepsilon} \frac{\varphi(\tau)}{(\tau - t)^m} d\tau - 2i \sum_{k=0}^{p-1} \frac{\left[\varphi\left(e^{ix}\right) e^{ix} \left(\dfrac{x - x_0}{e^{ix} - e^{ix_0}} \right)^m \right]^{(2k+1)} (x_0)}{(2k)!(2p - 2k - 1)[\delta(\varepsilon)]^{2p-2k-1}} \right)$$

when $m = 2p+1$, $p \in N$,

where $\gamma_\varepsilon = \{\tau \in \gamma_0 : |\tau - t| > \varepsilon\}$, $t = e^{ix_0}$, $\delta(\varepsilon) = 2\arcsin\frac{\varepsilon}{2}$, then the value of this limit is referred to as the hypersingular integral of the function $\dfrac{\varphi(\tau)}{(\tau - t)^m}$ on the unit circle γ_0 and is denoted by $\displaystyle\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau - t)^m} d\tau$.

From definitions 2 and 4, and the following relation

$$\lim_{\varepsilon \to 0+} \left(\frac{1}{\varepsilon} - \frac{1}{\delta(\varepsilon)} \right) = 0,$$

it follows,

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau - t)^m} d\tau = \int_{x_0-\pi}^{x_0+\pi} \frac{\varphi\left(e^{ix}\right) ie^{ix}}{\left(e^{ix} - e^{ix_0}\right)^m} dx, \tag{13}$$

where $t = e^{ix_0}$.

The formula (13) shows that, by means of the substitution $t = e^{ix}$ hypersingular integrals on the unit circle are reduced to the hypersingular integrals on an interval.

Theorem 2. If the function φ has absolutely continuous $(m-2)^{\text{th}}$ derivative on γ_0, then the hypersingular integral (12) exists for almost all $t \in \gamma_0$, and the following relation holds:

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau - t)^m} d\tau = \frac{1}{m-1} \int_{\gamma_0} \frac{\varphi'(\tau)}{(\tau - t)^{m-1}} d\tau. \tag{14}$$

Proof: The existence of the hypersingular integral (12) follows from (13). Let's prove equality (14). From (4), (5) and (13) it follows that,

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau - t)^m} d\tau - \frac{1}{m-1} \int_{\gamma_0} \frac{\varphi'(\tau)}{(\tau - t)^{m-1}} d\tau =$$

$$= \int_{x_0-\pi}^{x_0+\pi} \frac{\varphi\left(e^{ix}\right) ie^{ix}}{\left(e^{ix} - e^{ix_0}\right)^m} dx - \frac{1}{m-1} \int_{x_0-\pi}^{x_0+\pi} \frac{\varphi'\left(e^{ix}\right) ie^{ix}}{\left(e^{ix} - e^{ix_0}\right)^{m-1}} dx =$$

$$= \int_{x_0-\pi}^{x_0+\pi} \frac{\varphi\left(e^{ix}\right) ie^{ix}}{\left(e^{ix} - e^{ix_0}\right)^m} dx - \frac{1}{m-1} \int_{x_0-\pi}^{x_0+\pi} \frac{d\left(\varphi\left(e^{ix}\right)\right)}{\left(e^{ix} - e^{ix_0}\right)^{m-1}} =$$

$$= \int_{x_0-\pi}^{x_0+\pi} \frac{\varphi\left(e^{ix}\right) ie^{ix}}{\left(e^{ix} - e^{ix_0}\right)^m} dx - \frac{1}{m-1} \left[\left. \frac{\varphi\left(e^{ix}\right)}{\left(e^{ix} - e^{ix_0}\right)^{m-1}} \right|_{x_0-\pi}^{x_0+\pi} - \int_{x_0-\pi}^{x_0+\pi} \varphi\left(e^{ix}\right) d\left(\frac{1}{\left(e^{ix} - e^{ix_0}\right)^{m-1}} \right) \right] =$$

$$= \int_{x_0-\pi}^{x_0+\pi} \frac{\varphi\left(e^{ix}\right) ie^{ix}}{\left(e^{ix} - e^{ix_0}\right)^m} dx - \frac{1}{m-1} \left[0 + (m-1) \int_{x_0-\pi}^{x_0+\pi} \varphi\left(e^{ix}\right) \cdot \frac{ie^{ix}}{\left(e^{ix} - e^{ix_0}\right)^m} dx \right] = 0.$$

Theorem has been proved.

Corollary 1. If the function φ has absolutely continuous $(m-2)^{\text{th}}$ derivative on γ_0, then the the following relation holds:

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau-t)^m} d\tau = \frac{1}{(m-1)!} \int_{\gamma_0} \frac{\varphi^{(m-1)}(\tau)}{\tau - t} d\tau, \qquad (15)$$

where the integral standing on the right-hand side is understood in the sense of the Cauchy principal value.

Corollary 2. If we take into consideration $\varphi(\tau)=\tau^k$, $k=\overline{0,m-2}$ in (15), then we get

$$\int_{\gamma_0} \frac{\tau^k}{(\tau-t)^m} d\tau = 0, \quad k=\overline{0,m-2}.$$

Corollary 3. If the function φ is differentiable $(m-1)$ times at the point t, then the following equality is true:

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau-t)^m} d\tau = \int_{\gamma_0} \frac{\varphi(\tau) - \sum_{k=0}^{m-2} \frac{\varphi^{(k)}(t)}{k!}(\tau-t)^k}{(\tau-t)^m} d\tau,$$

where the integral standing on the right-hand side is understood in the sense of the Cauchy principal value.

4. Conclusion

In this paper, we have introduced different methods in order to define for hypersingular integrals of types with optimal accuracy. The proposed formulas can be applied to solve real-world engineering problems and have various successful applications in numerical implementations.

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