2025. Volume 9, Number 2 Pages 152-156

UDC:517.54

DOI: https://doi.org/10.30546/09090.2025.510.1029

SUFFICIENT CONDITIONS FOR A GENEREAL INTEGRAL OPERATOR RELATED WİTH BOUNDED BOUNDARY ROTATION

Erhan DENİZ*

Kafkas University,

edeniz36@gmail.com

Kars-Türkiye

Sercan KAZIMOĞLU

Kafkas University,

sercan.kazimoglu@kafkas.edu.tr

Kars-Türkiye

ARTICLE INFO

Article history
Received:2026-01-09
Received in revised form:2026-01-12
Accepted:2026-01-16
Available online

Keywords:
Analytic function;
Univalent functions;
Starlike and convex;
Integral operator.
2010 Mathematics Subject Classifications: 30C45

ABSTRACT

In this paper, we define a new subclass $\mathrm{SP}_k^{\lambda}(\delta,\alpha,\omega,\beta,m)$ *of analytic functions by means of an appropriate differential operator. The introduced class is characterized through analytic conditions involving several real and complex parameters. For functions belonging to this subclass, we investigate and determine various structural and functional properties. In particular, attention is focused on the integral operator* $I_m(f_1,\dots,f_r)$ *associated with this class of functions. Several results describing the behavior and preservation properties of this operator are established. The obtained findings contribute to the theory of analytic function classes defined via differential and integral operators and extend related results in the existing literature.*

1. Introduction

Let A denote the class of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (33)$$

defined in the open unit disc $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A containing univalent functions defined in U. Let $\mathrm{P}_k^{\lambda}(\beta)$ denote the class of analytic functions $p(z)$ in U satisfying the following properties:

i. $p(0) = 1$

ii. $\int_0^{2\pi} \left| \dfrac{\Re e^{i\lambda} p(z) - \beta\cos\lambda}{1-\beta} \right| d\theta \le k\pi \cos\lambda,$

where $k \geq 2$, λ is real, $|\lambda| < \frac{\pi}{2}$, $0 \leq \beta < 1$, $z = re^{i\theta}$, $0 \leq r < 1$.

Let $V_k^\lambda(\beta)$ (see [8]) denote the class of functions $f(z)$ analytic in U satisfies the normalization properties $f(0) = f'(0) - 1 = 0$ and

$$1 + \frac{z\, f''(z)}{f'(z)} \in P_k^\lambda(\beta),$$

where k, λ and β are as above.

For $\beta = 0$ we get the class V_k^λ of functions with bounded boundary rotation studied by Moulis [7].

Any function $f(z) \in V_k^\lambda(\beta)$ if and only if

$$\Re\left\{e^{i\lambda}\left(1 + \frac{z\, f''(z)}{f'(z)}\right)\right\} > \beta \cos\lambda, \qquad |z| < \frac{k - \sqrt{k^2 - 4}}{2}.$$

A function $f \in A$ with the normalization properties $f(0) = f'(0) - 1 = 0$ is said to be in the class $U_k^\lambda(\beta)$ if $\frac{z\, f'(z)}{f(z)} \in P_k^\lambda(\beta)$.

In [6], Darus and Faisal defined the differential operator $D_\delta^m(\alpha,\omega)$ as follows:

$$D_\delta^0(\alpha,\omega) f(z) = f(z),$$

$$D_\delta^1(\alpha,\omega) f(z) = Df(z) = \left(1 - \delta\omega^\alpha\right) f(z) + \delta\omega^\alpha z f'(z), \tag{2}$$

$$D_\delta^2(\alpha,\omega) f(z) = D\left(D_\delta^1(\alpha,\omega) f(z)\right),$$

$$\vdots$$

$$D_\delta^m(\alpha,\omega) f(z) = D\left(D_\delta^{m-1}(\alpha,\omega) f(z)\right),$$

where $\delta, \alpha, \omega \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

If f is given (1) then from the definition of the operator $D_\delta^m(\alpha,\omega) f(z)$ it is easy to see that

$$D_\delta^m(\alpha,\omega) f(z) = z + \sum_{n=2}^{\infty} \left[1 + (n-1)\delta\omega^\alpha\right]^m a_n z^n. \tag{3}$$

It should be remarked that the $D_\delta^m(\alpha,\omega)$ is a generalization of many other linear operators considered earlier. In particular, for $f \in A$ we have the following:

1. $D_1^m(1,1) f(z) = D^m f(z)$ the operator investigated by Salagean [9]
2. $D_\delta^m(1,1) f(z) = D_\delta^m f(z)$ the operator studied by Al-Oboudi [1]
3. $D_{1/2}^m(1,1) f(z) = I^m f(z)$ the operator studied by Uralegaddi and Somanatha [10].

Now, by making use of $D_\delta^m(\alpha,\omega)$, we define a new subclass of analytic functions.

Let $\mathrm{SP}_k^\lambda(\delta,\alpha,\omega,\beta,m)$ the class of functions $f\in \mathrm{A}$ and satisfying the condition

$$\frac{z\left(D_\delta^m(\alpha,\omega)f(z)\right)'}{D_\delta^m(\alpha,\omega)f(z)}\in \mathrm{P}_k^\lambda(\beta),$$

where β is real number with $0\le\beta<1$.

Let $r,m\in \mathbb{N}_0$ and $a_i>0.$ We define the integral operator $I_m : A^r\to A$

$$I_m(f_1,\dots,f_r)(z)=\int_0^z\left(\frac{D_\delta^m(\alpha,\omega)f_1(t)}{t}\right)^{a_1}\cdots\left(\frac{D_\delta^m(\alpha,\omega)f_r(t)}{t}\right)^{a_r}dt,\quad z\in \mathrm{U},\qquad (4)$$

where $f_i\in \mathrm{A}$.

For special value of $m=0$ we have the integral operator

$$I_0(f_1,\dots,f_r)(z)=\int_0^z\left(\frac{f_1(t)}{t}\right)^{a_1}\cdots\left(\frac{f_r(t)}{t}\right)^{a_r}dt \qquad (5)$$

introduced in [4].

2. Main Results

Our first result as follows.

Theorem 2.1. Let $a_i>0,$ and β_i be real numbers with the property $0\le\beta_i<1$ and $f_i\in \mathrm{SP}_k^\lambda(\delta,\alpha,\omega,\beta_i,m)$ for $i\in \mathbb{N}$. If $0<\sum_{i=1}^r a_i(1-\beta_i)\le 1,$ then $I_m(f_1,\cdots,f_r)\in \mathrm{V}_k^\lambda(\gamma)$, where $\gamma=1+\sum_{i=1}^m a_i(\beta_i-1).$

Proof. From (3), for $1\le i\le r,$ we have

$$\frac{D_\delta^m(\alpha,\omega)f_i(z)}{z}=1+\sum_{n=2}^\infty\left[1+(n-1)\delta\omega^\alpha\right]^m a_n z^{n-1},\quad m\in \mathbb{N}_0$$

and

$$\frac{D_\delta^m(\alpha,\omega)f_i(z)}{z}\neq 0,\quad \forall z\in \mathrm{U}.$$

We consider the operator

$$I_m(f_1,\dots,f_r)(z)=\int_0^z\left(\frac{D_\delta^m(\alpha,\omega)f_1(t)}{t}\right)^{a_1}\cdots\left(\frac{D_\delta^m(\alpha,\omega)f_r(t)}{t}\right)^{a_r}dt.$$

On successive differentiation of $I_m(f_1,\cdots,f_r)$ we get

$$I_m(f_1,\cdots,f_r)'(z)=\left(\frac{D_\delta^m(\alpha,\omega)f_1(z)}{z}\right)^{a_1}\cdots\left(\frac{D_\delta^m(\alpha,\omega)f_r(z)}{z}\right)^{a_r},$$

$$I_m\left(f_1,\cdots,f_r\right)''(z)=\sum_{i=1}^{r}a_i\left(\frac{D_\delta^m(\alpha,\omega)f_i(z)}{z}\right)^{a_i-1}\frac{z\left(D_\delta^m(\alpha,\omega)f_i(z)\right)'-D_\delta^m(\alpha,\omega)f_i(z)}{z^2}$$
$$\times\prod_{j=1}^{r}\left(\frac{D_\delta^m(\alpha,\omega)f_j(z)}{z}\right)^{a_j}$$

and so

$$\frac{I_m\left(f_1,\cdots,f_r\right)''(z)}{I_m\left(f_1,\cdots f_r\right)'(z)}=\sum_{i=1}^{r}a_i\left[\frac{z\left(D_\delta^m(\alpha,\omega)f_i(z)\right)'}{D_\delta^m(\alpha,\omega)f_i(z)}-\frac{1}{z}\right].$$

Thus we obtain that

$$e^{i\lambda}\left(1+\frac{z\,I_m\left(f_1,\cdots,f_r\right)''(z)}{I_m\left(f_1,\cdots,f_r\right)'(z)}\right)=\sum_{i=1}^{n}a_ie^{i\lambda}\left[\frac{z\left(D_\delta^m(\alpha,\omega)f_i(z)\right)'}{D_\delta^m(\alpha,\omega)f_i(z)}\right]+e^{i\lambda}\left(1-\sum_{i=1}^{r}a_i\right).$$

From the last equality, we have

$$\Re\left\{e^{i\lambda}\left(1+\frac{z\,I_m\left(f_1,\cdots,f_r\right)''(z)}{I_m\left(f_1,\cdots,f_r\right)'(z)}\right)\right\}=\sum_{i=1}^{r}a_i\Re\left\{e^{i\lambda}\left[\frac{z\left(D_\delta^m(\alpha,\omega)f_i(z)\right)'}{D_\delta^m(\alpha,\omega)f_i(z)}\right]\right\}+\cos\lambda\left(1-\sum_{i=1}^{r}a_i\right).$$

Since $f_i\in \mathrm{SP}_k^\lambda\left(\delta,\alpha,\omega,\beta_i,m\right)$ from the hypothesis of theorem we get

$$\Re\left\{e^{i\lambda}\left(1+\frac{z\,I_m\left(f_1,\cdots,f_r\right)''(z)}{I_m\left(f_1,\cdots,f_r\right)'(z)}\right)\right\}=\sum_{i=1}^{r}a_i\Re\left\{e^{i\lambda}\left[\frac{z\left(D_\delta^m(\alpha,\omega)f_i(z)\right)'}{D_\delta^m(\alpha,\omega)f_i(z)}\right]\right\}+\cos\lambda\left(1-\sum_{i=1}^{r}a_i\right)$$
$$>\sum_{i=1}^{r}\beta_ia_i\cos\lambda+\cos\lambda\left(1-\sum_{i=1}^{r}a_i\right)$$
$$=\cos\lambda\left(1+\sum_{i=1}^{r}a_i(\beta_i-1)\right).$$

Hence $I_m\left(f_1,\cdots,f_r\right)(z)\in \mathrm{V}_k^\lambda(\gamma)$, where $\gamma=1+\sum_{i=1}^{r}a_i\left(\beta_i-1\right)$.

For special values $m=0$, $k=2$ and $\lambda=0$ we get the following results.

Corollary 2.2. Let $a_i, i\in\{1,2,\cdots,r\}$ be real numbers with the properties $a_i>0$ for $i\in\{1,2,\cdots,r\}$ and $r<\sum_{i=1}^{r}a_i\leq r+1$. We suppose that the functions f_i are the starlike functions of order $\frac{1}{a_i}, i\in\{1,2,\cdots,r\}$, that is, $f_i\in \mathrm{S}^*\left(\frac{1}{a_i}\right)$. Then the integral operator defined in (4) is convex of order $\tilde{\gamma}=r+1-\sum_{i=1}^{r}a_i$ (see [2]).

Corollary 2.3. Let $a_i, i\in\{1,2,\cdots,r\}$ be real numbers with the properties $a_i>0$ for

$i\in\{1,2,\cdots,r\}$ and $0<\sum_{i=1}^{r}a_i\le 1$. We consider that the functions $f_i\in S^*$ for $i\in\{1,2,\cdots,r\}$. In these conditions, the integral operator defined in (4) is convex of order $1-\sum_{i=1}^{r}a_i$ (see [5]).

For $\beta_1=\beta_2=\cdots=\beta_r=\beta$ and $m=0$, similarly we prove the following theorem.

Theorem 2.4. Let a_i be real numbers with the properties $a_i>0$ and $f_i\in SP_k^{\lambda}(\delta,\alpha,\omega,\beta,0)$ for $i\in$ □ . If $0<\sum_{i=1}^{r}a_i\le\frac{1}{1-\beta}$, then the integral operator (4) is in the class $V_k^{\lambda}(\gamma)$, where $\gamma=(\beta-1)\sum_{i=1}^{r}a_i+1$.

For special values $m=0$, $k=2$ and $\lambda=0$ in Theorem 2.4, we get the following result [3].

Corollary 2.5. Let $a_i>0$ for $i\in\{1,2,\cdots,r\}$ and $f_i\in SP_2^{0}(\delta,\alpha,\omega,\beta,0)\,(0\le\beta<1)$. If $0<\sum_{i=1}^{r}a_i\le\frac{1}{1-\beta}$, then the integral operator $I_0(f_1,\cdots,f_r)$ is convex of order $(\beta-1)\sum_{i=1}^{r}a_i+1$.

3. Conclusion

In the current study, a new subclass of analytic functions associated with a differential operator has been introduced and investigated. Also we introduced a new general integral operator $I_m(f_1,\cdots,f_r)$. Main results related with sufficient conditons of this integral operator to belonging bounded boundary rotation. For special cases of parameters we obtained the earlier results.

REFERENCE LIST

[1] Al-Oboudi, F. M. (2004). On univalent functions defined by a generalized Salagean operator, *Int. J. Math. Math. Sci.*, 27, 1429-1436.

[2] Breaz, D. (2008). A convexity property for an integral operator on the class $S_p(\beta)$, *J. Inequal. Appl.*, 2008, Article ID 143869, 4 pages.

[3] Breaz, D., Breaz., N. (2006). Some convexity properties for a general integral operator, *J. Inequal. Pure Appl. Math.*, 7(5).

[4] Breaz, D., Breaz., N. (2002). Two integral operators, *Studia Universitatis Babeş -Bolyai, Mathematica*, Cluj Napoca, 47(3), 13-21.

[5] Breaz, D., Owa, S., Breaz, N. (2008). A new integral univalent operator, *Acta Univ. Apulensis Math. Inform.*, 16, 11-16.

[6] Darus, M., Faisal, I. (2012). Some subclasses of analytic functions of complex order defined by new differential operator. *Tamkang J. Math.*, 43(2), 223-242.

[7] Moulis, E. J. (1972). A generalization of univalent functions with bounded boundary rotation, *Trans. Am. Math. Soc.*, 174, 369 -381.

[8] Moulis, E. J. (1979). Generalization of the Robertson functions, *Pacific J. Math.*, 81, 169-174.

[9] Salagean, G.S. (1983). Subclasses of univalent functions, *Lect. Notes Math.*, Springer-Verlag, 1013, 362-372.

[10] Uralegaddi, B.A., Somanatha, C. (1992). Certain classes of univalent functions, *In: Current Topics in AnalyticFunction Theory. Eds. H.M. Srivastava and S. Owa. World Scientific Publishing Company, Singapore*, 3